

Rotations and Kinematics in State Estimation

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I Preliminaries

This document is a compiled set of notes on rotational representations and kinematics, synthesized from various textbooks, journal articles, and conference papers. Some sections closely follow the wording and derivations of the original sources, while others reflect my own exposition. Many equations and derivations are reproduced verbatim where clarity or completeness warrants it.

Although numerous high-quality references exist on rotations and kinematics of rigid bodies, most suffer from either rotation conventions without clear identification of the frames being related or unstated assumptions, which complicates cross-referencing across the literature. I attempted to address these issues with explicit derivations to provide a coherent bottom-up narrative. The goal is to offer a comprehensive overview of rotational representations and kinematics, exploring their interconnections and presenting each viewpoint from multiple angles.

1 Terminologies and Notations

Kinematics is the study of motion without reference to its causes, and consists of *translational* kinematics and *rotational* kinematics. Translational kinematics describes the motion of a rigid body through its position, velocity, and acceleration, while rotational kinematics describes the motion through its orientation, angular velocity, and angular acceleration.

1.1 Points and Bodies

A *point* represents a specific location or a position in three-dimensional space. It has zero size and zero mass. A *point mass* has zero size and nonnegative mass. Therefore, a point can be viewed as a point mass with zero mass [Ber18].

Points and point masses have position *relative* to other points and point masses. A *reference point* (e.g., origin of a frame) is a point that is relative to which the positions of other points are determined. Any point can be used as a reference point. Points and point masses can have *translational motion* relative to other points and point masses. Translational motion includes velocity and acceleration.

Points are denoted as single letters, e.g., x and y . Points that describe origins of frames are denoted with capital letters, e.g., A and B for frames F_A and F_B , respectively.

A *rigid body* is a finite collection of point masses and rigid massless links and joints whose shape does not change. In kinematics, objects are often treated as rigid bodies. Rigid bodies in three-dimensional space are typically free to translate and rotate. Mathematically, they have six degrees of freedom: three in translation and three in rotation. This six degree-of-freedom geometric configuration is known as the *pose* (position and orientation) of the rigid body [Bar24].

Since all motion is relative, position, velocity, and acceleration have meaning only with respect to a reference frame. In other words, a location of a point is meaningful only when given in terms of other points. A point or point mass cannot be “stationary” in absolute terms; in fact, the location of a point or point mass can be “fixed” only relative to other points and point masses. Analogous statements apply to rigid bodies under translation and rotation.

1.2 Vectors and Frames

1.2.1 Physical Vectors

A *physical vector* is a geometric object that has magnitude (or length) and direction. It does not have location, and therefore it is invariant under translation. In three-dimensional space, the set of all physical vectors forms a real vector space \mathbb{V} with $\dim(\mathbb{V}) = 3$. This space is *coordinate-free* (or *frame-independent*).

Bold letters with arrows are used to denote three-dimensional physical vectors, e.g., $\vec{\mathbf{x}}, \vec{\mathbf{y}} \in \mathbb{V}$. Since \mathbb{V} is a real vector space, algebraic operations such as addition, scalar multiplication, inner product (denoted by \cdot), and cross product (denoted by \times) are well-defined:

$$\begin{aligned} \vec{\mathbf{x}} + \vec{\mathbf{y}} &= \vec{\mathbf{z}}, & \vec{\mathbf{x}}, \vec{\mathbf{y}}, \vec{\mathbf{z}} &\in \mathbb{V}, \\ c\vec{\mathbf{x}} &= \vec{\mathbf{y}}, & \vec{\mathbf{x}}, \vec{\mathbf{y}} &\in \mathbb{V}, c \in \mathbb{R}, \\ \vec{\mathbf{x}} \cdot \vec{\mathbf{y}} &= d, & \vec{\mathbf{x}}, \vec{\mathbf{y}} &\in \mathbb{V}, d \in \mathbb{R}, \\ \vec{\mathbf{x}} \times \vec{\mathbf{y}} &= \vec{\mathbf{z}}, & \vec{\mathbf{x}}, \vec{\mathbf{y}}, \vec{\mathbf{z}} &\in \mathbb{V}. \end{aligned} \tag{1}$$

It is very important to note that the concept of a physical vector is *independent* of a frame. Operations such as transpose are not defined since it does not have any components until it is resolved in a frame.

It is useful to define a zero-physical vector denoted by $\vec{\mathbf{0}}$ which has magnitude zero, but no specified direction. Unit-length physical vectors that are orthogonal to each other are denoted as $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}} \in \mathbb{V}$ as discussed next.

1.2.2 Frames

A *frame* is a set of three physical vectors $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$, called the *basis vectors* or *axes*; unit length, dimensionless, linearly independent, and mutually perpendicular physical vectors arranged in a right-handed fashion. Each axis may be referred to as 3-axis, 2-axis, and 1-axis for $\hat{\mathbf{k}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{i}}$, respectively.

Under the right-hand rule, the basis vectors satisfy:

$$\begin{aligned} \hat{\mathbf{i}} \times \hat{\mathbf{j}} &= \hat{\mathbf{k}}, & \hat{\mathbf{j}} \times \hat{\mathbf{k}} &= \hat{\mathbf{i}}, & \hat{\mathbf{k}} \times \hat{\mathbf{i}} &= \hat{\mathbf{j}}, \\ \hat{\mathbf{i}} \cdot \hat{\mathbf{i}} &= \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1, \\ \hat{\mathbf{i}} \cdot \hat{\mathbf{j}} &= \hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0. \end{aligned} \tag{2}$$

Let F_A and F_B be frames. The basis vectors of F_A and F_B are denoted as $\{\hat{\mathbf{i}}_A, \hat{\mathbf{j}}_A, \hat{\mathbf{k}}_A\}$ and $\{\hat{\mathbf{i}}_B, \hat{\mathbf{j}}_B, \hat{\mathbf{k}}_B\}$, respectively.

A *vectrix* of a frame is a matrix formed by its basis vectors. A *row vectrix* and *column vectrix* are defined as:

$$F_A := \begin{bmatrix} \hat{\mathbf{i}}_A & \hat{\mathbf{j}}_A & \hat{\mathbf{k}}_A \end{bmatrix} \in \mathbb{V}^{1 \times 3}, \quad \mathcal{F}_A := F_A^T = \begin{bmatrix} \hat{\mathbf{i}}_A \\ \hat{\mathbf{j}}_A \\ \hat{\mathbf{k}}_A \end{bmatrix} \in \mathbb{V}^{3 \times 1}. \tag{3}$$

Vectrix can be scaled by a scalar $c \in \mathbb{R}$ as:

$$cF_A := \begin{bmatrix} c\hat{\mathbf{i}}_A & c\hat{\mathbf{j}}_A & c\hat{\mathbf{k}}_A \end{bmatrix} \in \mathbb{V}^{1 \times 3}, \quad c\mathcal{F}_A := \begin{bmatrix} c\hat{\mathbf{i}}_A \\ c\hat{\mathbf{j}}_A \\ c\hat{\mathbf{k}}_A \end{bmatrix} \in \mathbb{V}^{3 \times 1}. \tag{4}$$

This can be extended to matrix-vecatrix multiplication where $\mathbf{M} \in \mathbb{R}^{3 \times 3}$ acts on $\mathcal{F}_A \in \mathbb{V}^{3 \times 1}$:

$$\begin{aligned}
F_A \mathbf{M} &:= \begin{bmatrix} \hat{\mathbf{i}}_A & \hat{\mathbf{j}}_A & \hat{\mathbf{k}}_A \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} = \begin{bmatrix} M_{11} \hat{\mathbf{i}}_A + M_{21} \hat{\mathbf{j}}_A + M_{31} \hat{\mathbf{k}}_A \\ M_{12} \hat{\mathbf{i}}_A + M_{22} \hat{\mathbf{j}}_A + M_{32} \hat{\mathbf{k}}_A \\ M_{13} \hat{\mathbf{i}}_A + M_{23} \hat{\mathbf{j}}_A + M_{33} \hat{\mathbf{k}}_A \end{bmatrix}^T \in \mathbb{V}^{1 \times 3}, \\
\mathbf{M}^T \mathcal{F}_A &:= \begin{bmatrix} M_{11} & M_{21} & M_{31} \\ M_{12} & M_{22} & M_{32} \\ M_{13} & M_{23} & M_{33} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{i}}_A \\ \hat{\mathbf{j}}_A \\ \hat{\mathbf{k}}_A \end{bmatrix} = \begin{bmatrix} M_{11} \hat{\mathbf{i}}_A + M_{21} \hat{\mathbf{j}}_A + M_{31} \hat{\mathbf{k}}_A \\ M_{12} \hat{\mathbf{i}}_A + M_{22} \hat{\mathbf{j}}_A + M_{32} \hat{\mathbf{k}}_A \\ M_{13} \hat{\mathbf{i}}_A + M_{23} \hat{\mathbf{j}}_A + M_{33} \hat{\mathbf{k}}_A \end{bmatrix} \in \mathbb{V}^{3 \times 1}.
\end{aligned} \tag{5}$$

This is well-defined since it uses only scalar multiplication and addition of physical vectors, both of which are valid operations in \mathbb{V} .

The products between vecatrix with itself are defined as:

$$\begin{aligned}
F_A F_A^T &:= \begin{bmatrix} \hat{\mathbf{i}}_A & \hat{\mathbf{j}}_A & \hat{\mathbf{k}}_A \end{bmatrix} \begin{bmatrix} \hat{\mathbf{i}}_A \\ \hat{\mathbf{j}}_A \\ \hat{\mathbf{k}}_A \end{bmatrix} = \hat{\mathbf{i}}_A \cdot \hat{\mathbf{i}}_A + \hat{\mathbf{j}}_A \cdot \hat{\mathbf{j}}_A + \hat{\mathbf{k}}_A \cdot \hat{\mathbf{k}}_A = 3 \in \mathbb{R}, \\
F_A^T F_A &:= \begin{bmatrix} \hat{\mathbf{i}}_A \\ \hat{\mathbf{j}}_A \\ \hat{\mathbf{k}}_A \end{bmatrix} \begin{bmatrix} \hat{\mathbf{i}}_A & \hat{\mathbf{j}}_A & \hat{\mathbf{k}}_A \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{i}}_A \cdot \hat{\mathbf{i}}_A & \hat{\mathbf{i}}_A \cdot \hat{\mathbf{j}}_A & \hat{\mathbf{i}}_A \cdot \hat{\mathbf{k}}_A \\ \hat{\mathbf{j}}_A \cdot \hat{\mathbf{i}}_A & \hat{\mathbf{j}}_A \cdot \hat{\mathbf{j}}_A & \hat{\mathbf{j}}_A \cdot \hat{\mathbf{k}}_A \\ \hat{\mathbf{k}}_A \cdot \hat{\mathbf{i}}_A & \hat{\mathbf{k}}_A \cdot \hat{\mathbf{j}}_A & \hat{\mathbf{k}}_A \cdot \hat{\mathbf{k}}_A \end{bmatrix} = \mathbf{I} \in \mathbb{R}^{3 \times 3}.
\end{aligned} \tag{6}$$

Hence, we have the following vecatrix identities:

$$\begin{aligned}
F_A F_A^T &= F_A \mathcal{F}_A = \mathcal{F}_A^T \mathcal{F}_A = 3 \in \mathbb{R}, \\
F_A^T F_A &= \mathcal{F}_A F_A = \mathcal{F}_A \mathcal{F}_A^T = \mathbf{I} \in \mathbb{R}^{3 \times 3}.
\end{aligned} \tag{7}$$

The products between the vecrices of F_A and F_B are defined as:

$$\begin{aligned}
F_A F_B^T &:= \begin{bmatrix} \hat{\mathbf{i}}_A & \hat{\mathbf{j}}_A & \hat{\mathbf{k}}_A \end{bmatrix} \begin{bmatrix} \hat{\mathbf{i}}_B \\ \hat{\mathbf{j}}_B \\ \hat{\mathbf{k}}_B \end{bmatrix} = \hat{\mathbf{i}}_A \cdot \hat{\mathbf{i}}_B + \hat{\mathbf{j}}_A \cdot \hat{\mathbf{j}}_B + \hat{\mathbf{k}}_A \cdot \hat{\mathbf{k}}_B \in \mathbb{R}, \\
F_A^T F_B &:= \begin{bmatrix} \hat{\mathbf{i}}_A \\ \hat{\mathbf{j}}_A \\ \hat{\mathbf{k}}_A \end{bmatrix} \begin{bmatrix} \hat{\mathbf{i}}_B & \hat{\mathbf{j}}_B & \hat{\mathbf{k}}_B \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{i}}_A \cdot \hat{\mathbf{i}}_B & \hat{\mathbf{i}}_A \cdot \hat{\mathbf{j}}_B & \hat{\mathbf{i}}_A \cdot \hat{\mathbf{k}}_B \\ \hat{\mathbf{j}}_A \cdot \hat{\mathbf{i}}_B & \hat{\mathbf{j}}_A \cdot \hat{\mathbf{j}}_B & \hat{\mathbf{j}}_A \cdot \hat{\mathbf{k}}_B \\ \hat{\mathbf{k}}_A \cdot \hat{\mathbf{i}}_B & \hat{\mathbf{k}}_A \cdot \hat{\mathbf{j}}_B & \hat{\mathbf{k}}_A \cdot \hat{\mathbf{k}}_B \end{bmatrix} \in \mathbb{R}^{3 \times 3}.
\end{aligned} \tag{8}$$

We have the following identities:

$$\begin{aligned}
F_A F_B^T &= F_B F_A^T \\
&= \mathcal{F}_A^T \mathcal{F}_B = \mathcal{F}_B^T \mathcal{F}_A, \\
&= F_A \mathcal{F}_B = F_B \mathcal{F}_A \\
&= \hat{\mathbf{i}}_A \cdot \hat{\mathbf{i}}_B + \hat{\mathbf{j}}_A \cdot \hat{\mathbf{j}}_B + \hat{\mathbf{k}}_A \cdot \hat{\mathbf{k}}_B \in \mathbb{R},
\end{aligned} \tag{9}$$

and:

$$\begin{aligned}
F_A^T F_B = \mathcal{F}_A F_B = \mathcal{F}_A \mathcal{F}_B^T &= \begin{bmatrix} \hat{\mathbf{i}}_A \cdot \hat{\mathbf{i}}_B & \hat{\mathbf{i}}_A \cdot \hat{\mathbf{j}}_B & \hat{\mathbf{i}}_A \cdot \hat{\mathbf{k}}_B \\ \hat{\mathbf{j}}_A \cdot \hat{\mathbf{i}}_B & \hat{\mathbf{j}}_A \cdot \hat{\mathbf{j}}_B & \hat{\mathbf{j}}_A \cdot \hat{\mathbf{k}}_B \\ \hat{\mathbf{k}}_A \cdot \hat{\mathbf{i}}_B & \hat{\mathbf{k}}_A \cdot \hat{\mathbf{j}}_B & \hat{\mathbf{k}}_A \cdot \hat{\mathbf{k}}_B \end{bmatrix} \in \mathbb{R}^{3 \times 3}, \\
F_B^T F_A = \mathcal{F}_B F_A = \mathcal{F}_B \mathcal{F}_A^T &= \begin{bmatrix} \hat{\mathbf{i}}_B \cdot \hat{\mathbf{i}}_A & \hat{\mathbf{i}}_B \cdot \hat{\mathbf{j}}_A & \hat{\mathbf{i}}_B \cdot \hat{\mathbf{k}}_A \\ \hat{\mathbf{j}}_B \cdot \hat{\mathbf{i}}_A & \hat{\mathbf{j}}_B \cdot \hat{\mathbf{j}}_A & \hat{\mathbf{j}}_B \cdot \hat{\mathbf{k}}_A \\ \hat{\mathbf{k}}_B \cdot \hat{\mathbf{i}}_A & \hat{\mathbf{k}}_B \cdot \hat{\mathbf{j}}_A & \hat{\mathbf{k}}_B \cdot \hat{\mathbf{k}}_A \end{bmatrix} \in \mathbb{R}^{3 \times 3}.
\end{aligned} \tag{10}$$

Since a physical vector has no location, it cannot translate and thus, a frame has no position, velocity, or acceleration. However, a reference point can be associated with a frame called the *origin of the frame* and the frame is drawn as if all of the axes were located at the reference point.

Like any other point, the origin of a frame has position, velocity, and acceleration relative to other points, and can be used to define the translational kinematic properties (i.e., position, velocity, and acceleration) of other points in a relative sense. Velocities and acceleration depend on the frame with respect to which changes are observed. Hence, derivatives of physical vectors are defined only with respect to frames.

The origin of F_A is denoted as A . Frame F_A axes defines its orientation and its origin defines its position. Some authors refer to the pair (A, F_A) as a *coordinate frame*. In this document, regardless if F_A has an origin or not, it will be denoted as F_A .

1.2.3 Resolving Physical Vectors in Frames

Physical vectors exist independently of any reference frame. However, \mathbb{V} is *isomorphic* to \mathbb{R}^3 as a real vector space, i.e., $\mathbb{V} \cong \mathbb{R}^3$. This isomorphism is not unique. Choosing a frame F_A with basis vectors $\{\hat{\mathbf{i}}_A, \hat{\mathbf{j}}_A, \hat{\mathbf{k}}_A\}$ induces a bijective linear map $[\cdot]_A : \mathbb{V} \rightarrow \mathbb{V}^3$ which assigns each physical vector $\vec{\mathbf{x}} \in \mathbb{V}$ its *component vector* (or *coordinate vector*) \mathbf{x}^A in frame F_A :

$$[\vec{\mathbf{x}}]_A := \mathbf{x}^A := \begin{bmatrix} \hat{\mathbf{i}}_A \cdot \vec{\mathbf{x}} \\ \hat{\mathbf{j}}_A \cdot \vec{\mathbf{x}} \\ \hat{\mathbf{k}}_A \cdot \vec{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3, \tag{11}$$

where $x_1, x_2, x_3 \in \mathbb{R}$ are called the *components* of the physical vector $\vec{\mathbf{x}}$ resolved in F_A . This operation is called *resolving a physical vector in a frame* and expresses the physical vector in terms of the basis vectors of the frame.

The physical vector $\vec{\mathbf{x}}$ can be reconstructed from its component vector \mathbf{x}^A as:

$$\begin{aligned}
\vec{\mathbf{x}} &= (\hat{\mathbf{i}}_A \cdot \vec{\mathbf{x}}) \hat{\mathbf{i}}_A + (\hat{\mathbf{j}}_A \cdot \vec{\mathbf{x}}) \hat{\mathbf{j}}_A + (\hat{\mathbf{k}}_A \cdot \vec{\mathbf{x}}) \hat{\mathbf{k}}_A \\
&= x_1 \hat{\mathbf{i}}_A + x_2 \hat{\mathbf{j}}_A + x_3 \hat{\mathbf{k}}_A \\
&= \begin{bmatrix} \hat{\mathbf{i}}_A & \hat{\mathbf{j}}_A & \hat{\mathbf{k}}_A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
&= F_A \mathbf{x}^A = \mathcal{F}_A^T \mathbf{x}^A.
\end{aligned} \tag{12}$$

1.2.4 Resolving Vectrices in Frames

Let F_A and F_B be frames with axes $\{\hat{\mathbf{i}}_A, \hat{\mathbf{j}}_A, \hat{\mathbf{k}}_A\}$ and $\{\hat{\mathbf{i}}_B, \hat{\mathbf{j}}_B, \hat{\mathbf{k}}_B\}$, respectively. The resolution operation can be extended as $[\cdot]_A := \mathbb{V}^3 \rightarrow \mathbb{R}^{3 \times 3}$ to compute the *Gram Matrices*:

$$\begin{aligned}
[F_B]_A &:= \begin{bmatrix} [\hat{\mathbf{i}}_B]_A & [\hat{\mathbf{j}}_B]_A & [\hat{\mathbf{k}}_B]_A \end{bmatrix} \\
&= \begin{bmatrix} \hat{\mathbf{i}}_A \cdot \hat{\mathbf{i}}_B & \hat{\mathbf{i}}_A \cdot \hat{\mathbf{j}}_B & \hat{\mathbf{i}}_A \cdot \hat{\mathbf{k}}_B \\ \hat{\mathbf{j}}_A \cdot \hat{\mathbf{i}}_B & \hat{\mathbf{j}}_A \cdot \hat{\mathbf{j}}_B & \hat{\mathbf{j}}_A \cdot \hat{\mathbf{k}}_B \\ \hat{\mathbf{k}}_A \cdot \hat{\mathbf{i}}_B & \hat{\mathbf{k}}_A \cdot \hat{\mathbf{j}}_B & \hat{\mathbf{k}}_A \cdot \hat{\mathbf{k}}_B \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{i}}_B^A & \hat{\mathbf{j}}_B^A & \hat{\mathbf{k}}_B^A \end{bmatrix} \in \mathbb{R}^{3 \times 3}, \\
[\mathcal{F}_B]_A &:= \begin{bmatrix} [\hat{\mathbf{i}}_B]_A \\ [\hat{\mathbf{j}}_B]_A \\ [\hat{\mathbf{k}}_B]_A \end{bmatrix} = ([F_B]_A)^T \\
&= \begin{bmatrix} \hat{\mathbf{i}}_A \cdot \hat{\mathbf{i}}_B & \hat{\mathbf{j}}_A \cdot \hat{\mathbf{i}}_B & \hat{\mathbf{k}}_A \cdot \hat{\mathbf{i}}_B \\ \hat{\mathbf{i}}_A \cdot \hat{\mathbf{j}}_B & \hat{\mathbf{j}}_A \cdot \hat{\mathbf{j}}_B & \hat{\mathbf{k}}_A \cdot \hat{\mathbf{j}}_B \\ \hat{\mathbf{i}}_A \cdot \hat{\mathbf{k}}_B & \hat{\mathbf{j}}_A \cdot \hat{\mathbf{k}}_B & \hat{\mathbf{k}}_A \cdot \hat{\mathbf{k}}_B \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{i}}_A^B & \hat{\mathbf{j}}_A^B & \hat{\mathbf{k}}_A^B \end{bmatrix} \in \mathbb{R}^{3 \times 3}.
\end{aligned} \tag{13}$$

Each column of $[F_B]_A$ is a component vector obtained by resolving the basis vectors of frame F_B in frame F_A . Conversely, each column of $[\mathcal{F}_B]_A$ is the component vector obtained by resolving the basis vectors of frame F_A in frame F_B . Substituting (10) into above, we get:

$$\begin{aligned}
[F_B]_A &= F_A^T F_B = \mathcal{F}_A \mathcal{F}_B^T \in \mathbb{R}^{3 \times 3}, \\
[\mathcal{F}_B]_A &= F_B^T F_A = \mathcal{F}_B \mathcal{F}_A^T \in \mathbb{R}^{3 \times 3},
\end{aligned} \tag{14}$$

which implies:

$$[F_B]_A = ([\mathcal{F}_B]_A)^T. \tag{15}$$

A natural question is what happens if we resolve the vectrix of frame F_A in frame F_A itself. We have:

$$[F_A]_A = [\mathcal{F}_A]_A = \begin{bmatrix} \hat{\mathbf{i}}_A \cdot \hat{\mathbf{i}}_A & \hat{\mathbf{i}}_A \cdot \hat{\mathbf{j}}_A & \hat{\mathbf{i}}_A \cdot \hat{\mathbf{k}}_A \\ \hat{\mathbf{j}}_A \cdot \hat{\mathbf{i}}_A & \hat{\mathbf{j}}_A \cdot \hat{\mathbf{j}}_A & \hat{\mathbf{j}}_A \cdot \hat{\mathbf{k}}_A \\ \hat{\mathbf{k}}_A \cdot \hat{\mathbf{i}}_A & \hat{\mathbf{k}}_A \cdot \hat{\mathbf{j}}_A & \hat{\mathbf{k}}_A \cdot \hat{\mathbf{k}}_A \end{bmatrix} = \mathbf{I} \in \mathbb{R}^{3 \times 3}. \tag{16}$$

Hence, we have:

$$\hat{\mathbf{i}}_A^A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{j}}_A^A = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{k}}_A^A = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \hat{\mathbf{i}}_A^A, \hat{\mathbf{j}}_A^A, \hat{\mathbf{k}}_A^A \in \mathbb{R}^3. \tag{17}$$

1.3 Derivatives of Physical Vectors

The time derivative of a physical vector depends on the reference frame from which it is observed. Let $\vec{\mathbf{x}}(t)$ be a time-varying physical vector, and let F_A and F_B be reference frames. In general, an

observer attached to F_A will observe a different temporal evolution of $\vec{\mathbf{x}}(t)$ than an observer attached to F_B . Hence, the temporal derivative of a physical vector must always be specified with respect to a particular frame. Such derivatives are therefore called *frame derivatives*. The time derivative of $\vec{\mathbf{x}}(t)$ with respect to frame F_A (or as seen from frame F_A) is defined as:

$$\begin{aligned}\overset{A}{\dot{\vec{\mathbf{x}}}} &:= \lim_{\delta t \rightarrow 0} \frac{[\vec{\mathbf{x}}(t + \delta t) - \vec{\mathbf{x}}(t)]_A}{\delta t} \\ &= \lim_{\delta t \rightarrow 0} \frac{F_A \mathbf{x}^A(t + \delta t) - F_A \mathbf{x}^A(t)}{\delta t},\end{aligned}\tag{18}$$

where $\mathbf{x}^A(t) = [x_1(t), x_2(t), x_3(t)]^T \in \mathbb{R}^3$ is the component vector of $\vec{\mathbf{x}}$ in frame F_A . Since $\vec{\mathbf{x}}$ at time t can be expressed as:

$$\vec{\mathbf{x}}(t) = x_1(t)\hat{\mathbf{i}}_A + x_2(t)\hat{\mathbf{j}}_A + x_3(t)\hat{\mathbf{k}}_A,\tag{19}$$

the derivative can be rewritten as:

$$\overset{A}{\dot{\vec{\mathbf{x}}}} := \dot{x}_1\hat{\mathbf{i}}_A + \dot{x}_2\hat{\mathbf{j}}_A + \dot{x}_3\hat{\mathbf{k}}_A \in \mathbb{V},\tag{20}$$

where $\dot{x}_i := dx_i(t)/dt$. Note that $\hat{\mathbf{i}}_A = \hat{\mathbf{j}}_A = \hat{\mathbf{k}}_A = \mathbf{0}$ since the axes of F_A are constant with respect to F_A . Resolving $\overset{A}{\dot{\vec{\mathbf{x}}}}$ in frame F_A yields:

$$\begin{aligned}\overset{A}{[\dot{\vec{\mathbf{x}}}]_A} &= [\dot{x}_1\hat{\mathbf{i}}_A + \dot{x}_2\hat{\mathbf{j}}_A + \dot{x}_3\hat{\mathbf{k}}_A]_A \\ &= \dot{x}_1\hat{\mathbf{i}}_A^A + \dot{x}_2\hat{\mathbf{j}}_A^A + \dot{x}_3\hat{\mathbf{k}}_A^A \\ &= \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} \\ &= \dot{\mathbf{x}}^A,\end{aligned}\tag{21}$$

where $\dot{\mathbf{x}}^A$ is the time derivative of the component vector of $\vec{\mathbf{x}}$ in frame F_A as seen from frame F_A . Note that each component of \mathbf{x}^A is time-varying scalar and therefore, the derivative of a component vector with respect to a frame is just the time derivative of the components. Therefore, for an arbitrary frame F_B , we have:

$$\overset{B}{\dot{\mathbf{x}}^A} \equiv \overset{A}{\dot{\mathbf{x}}^A} \equiv \dot{\mathbf{x}}^A.\tag{22}$$

If a physical vector $\vec{\mathbf{x}}$ is scaled by a time-varying scalar $c(t)$, we have:

$$\overset{A}{c(t)\vec{\mathbf{x}}} = \dot{c}(t)\vec{\mathbf{x}} + c(t)\overset{A}{\dot{\vec{\mathbf{x}}}}.\tag{23}$$

Similarly, for sums of vectors, we have:

$$\overset{A}{\dot{\vec{\mathbf{x}}} + \dot{\vec{\mathbf{y}}}} = \overset{A}{\dot{\vec{\mathbf{x}}}} + \overset{A}{\dot{\vec{\mathbf{y}}}}.\tag{24}$$

The time derivatives of the basis vectors of frame F_A with respect to frame F_A is defined as:

$$\widehat{\dot{F}}_A := \begin{bmatrix} \widehat{\dot{\hat{i}}_A} & \widehat{\dot{\hat{j}}_A} & \widehat{\dot{\hat{k}}_A} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (25)$$

since the axes of F_A are constant with respect to F_A . General time derivatives of the basis vectors of one frame with respect to another frame is discussed in Section 9.

1.4 Common Vectors in Kinematics

Table 1 lists common vectors in kinematics that are used in this document.

Vector Type	Definition	Physical Vector (\mathbb{V})	Component Vector in F_A (\mathbb{R}^3)
Unit Dimensionless	Unit length and has no associated physical units	$\hat{\mathbf{x}}$	$\hat{\mathbf{x}}^A$
Dimensionless	Has no associated physical units	$\vec{\mathbf{x}} = \ \vec{\mathbf{x}}\ \hat{\mathbf{x}}$	\mathbf{x}^A
Frame Axis	Basis vectors of frame F_B	$\{\hat{\mathbf{i}}_B, \hat{\mathbf{j}}_B, \hat{\mathbf{k}}_B\}$	$\{\hat{\mathbf{i}}_B^A, \hat{\mathbf{j}}_B^A, \hat{\mathbf{k}}_B^A\}$
Unit Angle	Unit angle vector of the physical vector $\vec{\mathbf{y}}$ relative to the physical vector $\vec{\mathbf{x}}$. The direction is given by the right hand rule with the fingers curled from $\vec{\mathbf{x}}$ to $\vec{\mathbf{y}}$ through the short-way angle $\theta_{\vec{\mathbf{y}}\vec{\mathbf{x}}} = \theta_{\vec{\mathbf{x}}\vec{\mathbf{y}}}$ between $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$	$\hat{\theta}_{\vec{\mathbf{y}}\vec{\mathbf{x}}} = -\hat{\theta}_{\vec{\mathbf{x}}\vec{\mathbf{y}}}$	$\hat{\theta}_{\vec{\mathbf{y}}\vec{\mathbf{x}}}^A$
Angle	Angle vector of the physical vector $\vec{\mathbf{y}}$ relative to the physical vector $\vec{\mathbf{x}}$. The angle $\theta_{\vec{\mathbf{y}}\vec{\mathbf{x}}} = \theta_{\vec{\mathbf{x}}\vec{\mathbf{y}}} \in (0, \pi)$ is the short-way angle between $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ and the direction is $\hat{\theta}_{\vec{\mathbf{y}}\vec{\mathbf{x}}}$	$\vec{\theta}_{\vec{\mathbf{y}}\vec{\mathbf{x}}} = \theta_{\vec{\mathbf{y}}\vec{\mathbf{x}}}\hat{\theta}_{\vec{\mathbf{y}}\vec{\mathbf{x}}}$	$\theta_{\vec{\mathbf{y}}\vec{\mathbf{x}}}^A$
Angular Velocity	Angular velocity of the frame F_C relative to the frame F_B	$\vec{\omega}_{CB}$	ω_{CB}^A
Rotation	Rotation vector of frame F_C relative to frame F_B	$\vec{\phi}_{CB}$	ϕ_{CB}^A
Angular Acceleration	Angular acceleration of the frame F_B relative to the frame F_A as seen from frame F_A	${}^A\vec{\alpha}_{BA}$	${}^A\alpha_{BA}^A$
Position	Position of point y relative to point x	$\vec{\mathbf{p}}_{yx}$	\mathbf{p}_{yx}^A
Velocity	Velocity of point y relative to point x as seen from frame F_A	${}^A\vec{\mathbf{v}}_{yx}$	${}^A\mathbf{v}_{yx}^A$
Acceleration	Acceleration of point y relative to point x as seen from frame F_A	${}^A\vec{\mathbf{a}}_{yx}$	${}^A\mathbf{a}_{yx}^A$

Table 1: Common Vectors in Kinematics

II The Rotation Group $SO(3)$

In this part, we introduce the three-dimensional rotation group $SO(3)$ together with its algebraic structure and its nontrivial topology.

2 Algebraic Structure of $SO(3)$

In this section, we introduce rotation as a linear operator that acts on physical vectors, preserving their lengths and relative orientation. The collection of all such operators forms a non-commutative group under composition called the *special orthogonal group* $SO(3)$. This group structure exists independently of any reference frame or numerical representation.

2.1 Rotation Operation

A *rotation* is a linear operator $r : \mathbb{V} \rightarrow \mathbb{V}$ acting on physical vectors. Given a physical vector $\vec{x} \in \mathbb{V}$, the rotated physical vector $\vec{y} \in \mathbb{V}$ can be obtained as:

$$\boxed{\vec{y} = r(\vec{x})}. \quad (26)$$

The rotation operator satisfies the following properties:

1. *Linearity*:

$$r(a\vec{x} + b\vec{y}) = ar(\vec{x}) + br(\vec{y}), \quad \vec{x}, \vec{y} \in \mathbb{V}, \forall a, b \in \mathbb{R}. \quad (27)$$

2. *Preserves the physical vector norm (length)*:

$$\|r(\vec{x})\| = \sqrt{r(\vec{x}) \cdot r(\vec{x})} = \sqrt{\vec{x} \cdot \vec{x}} := \|\vec{x}\|, \quad \forall \vec{x} \in \mathbb{V}. \quad (28)$$

3. *Preserves the angles between physical vectors*:

$$r(\vec{x}) \cdot r(\vec{y}) = \vec{x} \cdot \vec{y} = \|\vec{x}\| \|\vec{y}\| \cos \alpha, \quad \forall \vec{x}, \vec{y} \in \mathbb{V}, \alpha \in \mathbb{R}. \quad (29)$$

4. *Preserves the relative orientations of physical vectors*:

$$\vec{x} \times \vec{y} = \vec{z} \iff r(\vec{x}) \times r(\vec{y}) = r(\vec{z}), \quad \forall \vec{x}, \vec{y}, \vec{z} \in \mathbb{V}. \quad (30)$$

The rotation operator is a *frame-free* geometric operator that is fully characterized by the properties above. When computing numerically, a frame and a specific rotation representation must be chosen.

2.2 Special Orthogonal Group, $SO(3)$

A *group* (G, \circ) is a set G together with a binary operation (denoted by \circ) on G that satisfies the following four requirements known as the *group axioms*:

1. *Closure*: $\forall g_1, g_2 \in G, g_1 \circ g_2 \in G$.
2. *Associativity*: $\forall g_1, g_2, g_3 \in G, g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3 = g_1 \circ g_2 \circ g_3$.
3. *Identity element*: $\exists g_0 \in G$ such that $\forall g \in G, g_0 \circ g = g \circ g_0 = g$.
4. *Inverse element*: $\forall g \in G, \exists g^{-1} \in G$ such that $g \circ g^{-1} = g_0$.

For example, the set of all integers \mathbb{Z} with the addition operator forms the group $(\mathbb{Z}, +)$.

The set of all rotations in three dimensional space with function composition \circ forms a group called the *special orthogonal group*, denoted $SO(3)$:

$$SO(3) := \{r : \mathbb{V} \rightarrow \mathbb{V} \mid r \text{ satisfies rotation operator properties}\}. \quad (31)$$

The group axioms are verified at the operator level as follows:

1. *Closure*: If $r_1, r_2 \in SO(3)$, then $r_1 \circ r_2 \in SO(3)$. The composition of two rotation operators that each preserve lengths, angles, and orientations also preserves lengths, angles, and orientations.
2. *Associativity*: $\forall r_1, r_2, r_3 \in SO(3)$, $r_1 \circ (r_2 \circ r_3) = (r_1 \circ r_2) \circ r_3$. This follows from the associativity of function composition.
3. *Identity element*: The identity rotation operator $r_0(\vec{x}) = \vec{x}$ satisfies $r_0 \circ r = r \circ r_0 = r$ for all $r \in SO(3)$.
4. *Inverse element*: For every $r \in SO(3)$, there exists $r^{-1} \in SO(3)$ such that $r \circ r^{-1} = r^{-1} \circ r = r_0$. Geometrically, r^{-1} is the inverse rotation of r .

A *non-abelian group* or *non-commutative group*, is a group (G, \circ) in which there exists at least one pair of elements g_1 and g_2 of G , such that $g_1 \circ g_2 \neq g_2 \circ g_1$. $SO(3)$ is non-commutative group. Therefore, we have to speak of composing r_1 by r_2 *on the right* or *on the left* since $r_1 \circ r_2 \neq r_2 \circ r_1$.

To see this, consider two successive 90° rotations of a rigid body about the fixed axes of a reference frame: a 90° rotation about the 1-axis followed by a 90° rotation about the 3-axis produces a different final orientation than performing them in the reverse order. In contrast, vectors in \mathbb{R}^n commute under addition. This non-commutativity is one of the fundamental reasons why rotations cannot be treated as elements of a vector space.

2.3 Matrix Realization of $SO(3)$

The elements of $SO(3)$ as defined in (31) are rotation operators $r : \mathbb{V} \rightarrow \mathbb{V}$ on the frame-free three-dimensional vector space \mathbb{V} , and the group operation is function composition. To perform numerical computations, the group must be *realized* in terms of concrete mathematical objects equipped with explicit group operations.

As discussed in Section 1.2.4, choosing a frame F_A with basis vectors $\{\hat{i}_A, \hat{j}_A, \hat{k}_A\}$ establishes a bijection $[\cdot]_A : \mathbb{V} \rightarrow \mathbb{R}^3$ that sends each physical vector $\vec{x} \in \mathbb{V}$ to its component vector $\mathbf{x}^A \in \mathbb{R}^3$. Under this bijection, the linear operator r on \mathbb{V} corresponds to a unique matrix $\mathbf{R} \in \mathbb{R}^{3 \times 3}$ defined by:

$$[r(\vec{x})]_A = \mathbf{R} [\vec{x}]_A = \mathbf{R} \mathbf{x}^A \quad \forall \vec{x} \in \mathbb{V}. \quad (32)$$

The matrix \mathbf{R} is called a *rotation matrix* and represents r in frame F_A . Let $\mathbf{x}^A = [x_1, x_2, x_3] \in \mathbb{R}^3$ such that $\vec{x} = x_1 \hat{i}_A + x_2 \hat{j}_A + x_3 \hat{k}_A$. We have:

$$\begin{aligned} [r(\vec{x})]_A &= \left[r \left(x_1 \hat{i}_A + x_2 \hat{j}_A + x_3 \hat{k}_A \right) \right]_A \\ &= \left[x_1 r(\hat{i}_A) + x_2 r(\hat{j}_A) + x_3 r(\hat{k}_A) \right]_A \\ &= \left[\begin{array}{ccc} r(\hat{i}_A) & r(\hat{j}_A) & r(\hat{k}_A) \end{array} \right]_A \mathbf{x}^A \\ &= \underbrace{[r(F_A)]_A}_{\mathbf{R}} \mathbf{x}^A, \end{aligned} \quad (33)$$

where we defined $r(F_A) := [r(\hat{i}_A), r(\hat{j}_A), r(\hat{k}_A)]$. Note that we used the distributive property of r since it is a linear operator. From (32) and (33), we have:

$$\mathbf{R} := \underbrace{[r(F_A)]_A}_1 \in \mathbb{R}^{3 \times 3}. \quad (34)$$

Equation (34) shows that the rotation matrix \mathbf{R} encodes how each basis vector of frame F_A transforms under the rotation operation r (denoted by 1), with the transformation result resolved in F_A (denoted by 2).

Clearly, not every matrix in $\mathbb{R}^{3 \times 3}$ represents a rotation. The subset of matrices that are orthogonal and have determinant +1 (i.e., *proper* matrices), equipped with matrix multiplication, forms $SO(3)$. Therefore, the *matrix representation* of $SO(3)$ is defined as:

$$\boxed{SO(3) = \{\mathbf{R} \in \mathbb{R}^{3 \times 3} \mid \mathbf{R}\mathbf{R}^T = \mathbf{I}, \det(\mathbf{R}) = 1\}}. \quad (35)$$

To see this, we can check if $SO(3)$ defined as (35) satisfies the group axioms:

1. *Closure*: $\forall \mathbf{R}_1, \mathbf{R}_2 \in SO(3), \mathbf{R}_1\mathbf{R}_2 \in SO(3)$.
2. *Associativity*: $\forall \mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3 \in SO(3), \mathbf{R}_1(\mathbf{R}_2\mathbf{R}_3) = (\mathbf{R}_1\mathbf{R}_2)\mathbf{R}_3 = \mathbf{R}_1\mathbf{R}_2\mathbf{R}_3$.
3. *Identity element*: $\mathbf{I} \in SO(3)$ such that $\forall \mathbf{R} \in SO(3), \mathbf{I}\mathbf{R} = \mathbf{R}\mathbf{I} = \mathbf{R}$.
4. *Inverse element*: $\forall \mathbf{R} \in SO(3), \exists \mathbf{R}^{-1} \in SO(3)$ such that $\mathbf{R}\mathbf{R}^{-1} = \mathbf{R}^{-1}\mathbf{R} = \mathbf{I}$, with $\mathbf{R}^{-1} = \mathbf{R}^T$.

Furthermore, we can check if $SO(3)$ representation by (35) satisfies the rotation operator properties defined in Section 2.1. Let F_A be a frame and let $\mathbf{x}^A, \mathbf{y}^A, \mathbf{z}^A \in \mathbb{R}^3$ be the component vectors of physical vectors $\vec{\mathbf{x}}, \vec{\mathbf{y}}, \vec{\mathbf{z}} \in \mathbb{V}$ resolved in F_A . Given a rotation matrix $\mathbf{R} \in SO(3)$:

1. *Linearity*:

$$\mathbf{R}(a\mathbf{x}^A + b\mathbf{y}^A) = a\mathbf{R}\mathbf{x}^A + b\mathbf{R}\mathbf{y}^A, \quad \mathbf{x}^A, \mathbf{y}^A \in \mathbb{R}^3, \quad a, b \in \mathbb{R}. \quad (36)$$

2. *Preserves the component vector norm*:

$$\|\mathbf{R}\mathbf{x}^A\| = \sqrt{(\mathbf{R}\mathbf{x}^A)^T \mathbf{R}\mathbf{x}^A} = \sqrt{(\mathbf{x}^A)^T \mathbf{R}^T \mathbf{R}\mathbf{x}^A} = \sqrt{(\mathbf{x}^A)^T \mathbf{x}^A} = \|\mathbf{x}^A\|, \quad \forall \mathbf{x}^A \in \mathbb{R}^3. \quad (37)$$

3. *Preserves the angle between component vectors*:

$$(\mathbf{R}\mathbf{x}^A)^T \mathbf{R}\mathbf{y}^A = (\mathbf{x}^A)^T \mathbf{R}^T \mathbf{R}\mathbf{y}^A = (\mathbf{x}^A)^T \mathbf{y}^A = \|\mathbf{x}^A\| \|\mathbf{y}^A\| \cos \alpha, \quad \forall \mathbf{x}^A, \mathbf{y}^A \in \mathbb{R}^3. \quad (38)$$

4. *Preserves the relative orientations of component vectors*:

$$\mathbf{x}^A \times \mathbf{y}^A = \mathbf{z}^A \iff \mathbf{R}\mathbf{x}^A \times \mathbf{R}\mathbf{y}^A = \mathbf{R}\mathbf{z}^A, \quad \forall \mathbf{x}^A, \mathbf{y}^A, \mathbf{z}^A \in \mathbb{R}^3. \quad (39)$$

2.4 Euler's Rotation Theorem

Euler's rotation theorem states that any orientation-preserving displacement of a rigid body with one point fixed is equivalent to a single rotation about some axis passing through that fixed point. In other words, every three-dimensional rotation can be parameterized by a unique *axis-angle* pair in three-dimensional space. The *angle of rotation* is denoted as $\theta \in [0, \pi]$ and the *unit axis of rotation* is denoted as $\vec{\mathbf{u}} \in \mathbb{V}$. The rotation operation described by the pair $(\vec{\mathbf{u}}, \theta)$ is called the *axis-angle* representation (Section 5). It is common to define a *rotation vector*:

$$\vec{\phi} := \vec{\mathbf{u}}\theta \quad \in \mathbb{V}. \quad (40)$$

We define $r\{\vec{\phi}\}$ as the rotation operation parameterized by $(\vec{\mathbf{u}}, \theta)$. $r\{\vec{\phi}\}$ rotates a physical vector by the angle θ following the right-hand rule along the axis of rotation $\vec{\mathbf{u}}$. Note that $\vec{\mathbf{u}} = \vec{\phi}/\|\vec{\phi}\|$ and $\theta = \|\vec{\phi}\|$.

Consider the rotation of $\vec{\mathbf{x}} \in \mathbb{V}$ with the rotation operator $r\{\vec{\phi}\}$ producing a rotated vector $\vec{\mathbf{y}} \in \mathbb{V}$ as illustrated in Figure 1 [Sol17]. The angle of rotation, $\theta \in \mathbb{R}$ is the angle between $\vec{\mathbf{y}}$ and $\vec{\mathbf{x}}$. The rotation is accomplished by decomposing the physical vector $\vec{\mathbf{x}}$ into a part parallel to $\vec{\mathbf{u}}$ (denoted as $\vec{\mathbf{x}}_{\parallel}$), and a part orthogonal to $\vec{\mathbf{u}}$ (denoted as $\vec{\mathbf{x}}_{\perp}$), such that:

$$\vec{\mathbf{x}} = \vec{\mathbf{x}}_{\parallel} + \vec{\mathbf{x}}_{\perp}, \quad (41)$$

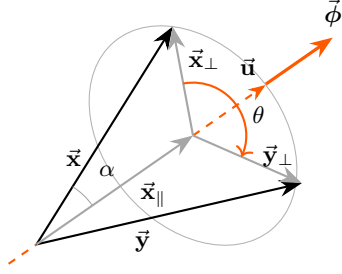


Figure 1: Rotation of a physical vector \vec{x} by an angle θ about the unit axis of rotation \vec{u} .

where:

$$\begin{aligned}\vec{x}_{\parallel} &= \vec{u} (\|\vec{x}\| \cos \alpha) = \vec{u} (\vec{u} \cdot \vec{x}), \\ \vec{x}_{\perp} &= \vec{x} - \vec{x}_{\parallel} = \vec{x} - \vec{u} (\vec{u} \cdot \vec{x}).\end{aligned}\quad (42)$$

The parallel part does not rotate after rotation, i.e., $\vec{y}_{\parallel} = \vec{x}_{\parallel}$ and the orthogonal part experiences a planar rotation in the plane normal to \vec{u} (shown as a gray ellipsoid). If we define a new frame F_E on this plane with basis vectors $\{\hat{i}_E, \hat{j}_E \in \mathbb{V} \mid \|\hat{i}_E\| = \|\hat{j}_E\| = 1\}$ such that:

$$\begin{aligned}\hat{i}_E &= \vec{x}_{\perp} \\ \hat{j}_E &= \vec{u} \times \vec{x}_{\perp} = \vec{u} \times \vec{x},\end{aligned}\quad (43)$$

we have $\vec{x}_{\perp} = \hat{i}_E \cdot 1 + \hat{j}_E \cdot 0$. A rotation of θ radians on this plane produces:

$$\begin{aligned}\vec{y}_{\perp} &= \hat{i}_E \cos \theta + \hat{j}_E \sin \theta \\ &= \vec{x}_{\perp} \cos \theta + (\vec{u} \times \vec{x}) \sin \theta.\end{aligned}\quad (44)$$

Hence, the rotated physical vector \vec{y} can be expressed as:

$$\begin{aligned}\vec{y} &= \vec{y}_{\parallel} + \vec{y}_{\perp} \\ &= \vec{x}_{\parallel} + \vec{x}_{\perp} \cos \theta + (\vec{u} \times \vec{x}) \sin \theta.\end{aligned}\quad (45)$$

Therefore, we get:

$$\boxed{\vec{y} = \vec{x} \cos \theta + (\vec{u} \times \vec{x}) \sin \theta + \vec{u} (\vec{u} \cdot \vec{x}) (1 - \cos \theta)},\quad (46)$$

which is called the *Rodrigues' rotation formula*.

The immediate consequence from the rotation formula is that the unit axis of rotation \vec{u} is *rotation invariant* under rotation by $\vec{\phi} = \theta \vec{u}$:

$$\begin{aligned}r(\vec{u}) &= \vec{u} \cos \theta + (\vec{u} \times \vec{u}) \sin \theta + \vec{u} (\vec{u} \cdot \vec{u}) (1 - \cos \theta) \\ &= \vec{u} \cos \theta + \mathbf{0} + \vec{u} - \vec{u} \cos \theta \\ &= \vec{u}.\end{aligned}\quad (47)$$

2.5 Duality of Rotations

Let $r : \mathbb{V} \rightarrow \mathbb{V}$ be a rotation operator acting on physical vector $\vec{x} \in \mathbb{V}$. There are two ways to describe a three-dimensional rotation: *active* (or *alibi*) and *passive* (or *alias*) rotations.

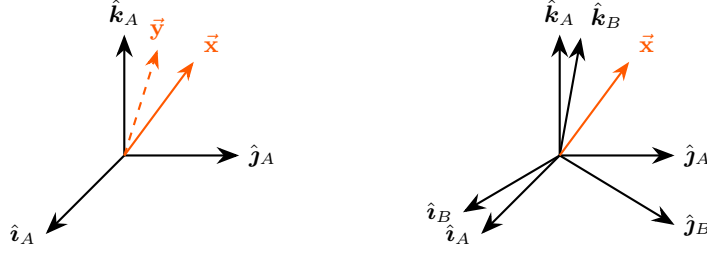


Figure 2: Duality of rotations: (left) Active rotation: Physical vector \vec{x} is rotated in frame F_A . (right) Passive rotation: Frame F_A vectrix is rotated to get frame F_B vectrix.

2.5.1 Active Rotations

2.5.1.1 Active Vector Rotation

In *active rotation*, the rotation operation r physically rotates a physical vector \vec{x} yielding a rotated vector \vec{y} :

$$\vec{y} = r(\vec{x}) \in \mathbb{V}. \quad (48)$$

If a reference frame F_A with axes $\{\hat{i}_A, \hat{j}_A, \hat{k}_A\}$ is fixed as illustrated in Figure 2 (left), the rotation operation can be described by the rotation matrix as defined in (34):

$$\begin{aligned} \mathbf{y}^A &= \mathbf{R}\mathbf{x}^A \\ &= \underbrace{[r(F_A)]_A}_{\mathbf{R}_{\text{active}}}\mathbf{x}^A, \end{aligned} \quad (49)$$

where $r(F_A)$ denotes the column-wise action of r on the vectrix F_A and $\mathbf{x}^A, \mathbf{y}^A \in \mathbb{R}^3$ are the component vectors of \vec{x}, \vec{y} in F_A , respectively. The rotation matrix in (49) is called an *active rotation matrix*:

$$\boxed{\mathbf{R}_{\text{active}} := \mathbf{R} = [r(F_A)]_A}. \quad (50)$$

For active rotation of a vector, its matrix representation requires only a single reference frame F_A . However, it is important to note that the basis vectors of frame F_A are rotated to generate a new *implicit frame* with rotated basis vectors.

2.5.1.2 Active Frame Rotation

Since frame axes are physical vectors, a frame vectrix can be rotated by r to create a new frame. This new frame is identical to the implicit frame generated by the rotated basis vectors in Section 2.5.1.1.

Let F_A be a frame with basis vectors $\{\hat{i}_A, \hat{j}_A, \hat{k}_A\}$. A new frame F_B with basis vectors $\{\hat{i}_B, \hat{j}_B, \hat{k}_B\}$ is obtained from F_A by the rotation operator r :

$$\begin{aligned} \hat{i}_B &= r(\hat{i}_A), \\ \hat{j}_B &= r(\hat{j}_A), \\ \hat{k}_B &= r(\hat{k}_A). \end{aligned} \quad (51)$$

The vectrix of the frame F_B is:

$$\begin{aligned} F_B &= \begin{bmatrix} r(\hat{i}_A) & r(\hat{j}_A) & r(\hat{k}_A) \end{bmatrix} \\ &= r(F_A). \end{aligned} \quad (52)$$

2.5.1.3 Active Rotation Matrix

Fundamentally, there is no difference between active vector rotation and active frame rotation. Substituting (52) into (50), we get:

$$\mathbf{R}_{\text{active}} = [r(F_A)]_A = [F_B]_A. \quad (53)$$

From (14), we have:

$$\begin{aligned} \mathbf{R}_{\text{active}} &= [F_B]_A = ([\mathcal{F}_B]_A)^T \\ &= F_A^T F_B = \mathcal{F}_A \mathcal{F}_B^T \\ &= \begin{bmatrix} \hat{\mathbf{i}}_B^A & \hat{\mathbf{j}}_B^A & \hat{\mathbf{k}}_B^A \end{bmatrix}. \end{aligned} \quad (54)$$

Hence, each column of $\mathbf{R}_{\text{active}}$ is the single basis vector of F_B resolved in F_A .

2.5.2 Passive Rotations

Let F_A be a frame and $\vec{\mathbf{x}} \in \mathbb{V}$ be a physical vector. In *passive rotation*, the rotation operation rotates the basis vectors of frame F_A , yielding a new frame F_B as illustrated in Figure 2 (right). The physical vector stays fixed during the rotation. Hence, it is a change of basis operation; effectively re-expressing the vector in the new reference frame F_B . Passive rotation is another way to look at the active frame rotation.

2.5.2.1 Passive Rotation Matrix

Let $x_1^A, x_2^A, x_3^A \in \mathbb{R}$ be the components of $\vec{\mathbf{x}}$ in F_A . The component vector of $\vec{\mathbf{x}}$ in frame F_B can be expressed as:

$$\mathbf{x}^B = [\vec{\mathbf{x}}]_B \quad (55)$$

$$= \begin{bmatrix} \hat{\mathbf{i}}_B \cdot \vec{\mathbf{x}} \\ \hat{\mathbf{j}}_B \cdot \vec{\mathbf{x}} \\ \hat{\mathbf{k}}_B \cdot \vec{\mathbf{x}} \end{bmatrix} \quad (56)$$

$$= \begin{bmatrix} \hat{\mathbf{i}}_B \cdot (x_1^A \hat{\mathbf{i}}_A + x_2^A \hat{\mathbf{j}}_A + x_3^A \hat{\mathbf{k}}_A) \\ \hat{\mathbf{j}}_B \cdot (x_1^A \hat{\mathbf{i}}_A + x_2^A \hat{\mathbf{j}}_A + x_3^A \hat{\mathbf{k}}_A) \\ \hat{\mathbf{k}}_B \cdot (x_1^A \hat{\mathbf{i}}_A + x_2^A \hat{\mathbf{j}}_A + x_3^A \hat{\mathbf{k}}_A) \end{bmatrix} \quad (57)$$

$$= \begin{bmatrix} \hat{\mathbf{i}}_B \cdot \hat{\mathbf{i}}_A & \hat{\mathbf{i}}_B \cdot \hat{\mathbf{j}}_A & \hat{\mathbf{i}}_B \cdot \hat{\mathbf{k}}_A \\ \hat{\mathbf{j}}_B \cdot \hat{\mathbf{i}}_A & \hat{\mathbf{j}}_B \cdot \hat{\mathbf{j}}_A & \hat{\mathbf{j}}_B \cdot \hat{\mathbf{k}}_A \\ \hat{\mathbf{k}}_B \cdot \hat{\mathbf{i}}_A & \hat{\mathbf{k}}_B \cdot \hat{\mathbf{j}}_A & \hat{\mathbf{k}}_B \cdot \hat{\mathbf{k}}_A \end{bmatrix} \begin{bmatrix} x_1^A \\ x_2^A \\ x_3^A \end{bmatrix} \quad (58)$$

$$= \underbrace{[F_A]_B}_{\mathbf{R}_{\text{passive}}} \mathbf{x}^A, \quad (59)$$

where $\mathbf{R}_{\text{passive}} := [F_A]_B$ is called the *passive rotation matrix*. From (14), we have:

$$\begin{aligned} \mathbf{R}_{\text{passive}} &= [F_A]_B = ([\mathcal{F}_A]_B)^T \\ &= F_B^T F_A = \mathcal{F}_B \mathcal{F}_A^T \\ &= \begin{bmatrix} \hat{\mathbf{i}}_A^B & \hat{\mathbf{j}}_A^B & \hat{\mathbf{k}}_A^B \end{bmatrix}. \end{aligned} \quad (60)$$

Hence, each column of $\mathbf{R}_{\text{passive}}$ is the single basis vector of F_A resolved in F_B .

2.5.3 Interpretation of Active and Passive Rotations

We have defined active and passive rotation matrices as:

$$\mathbf{R}_{\text{active}} = [F_B]_A = ([\mathcal{F}_B]_A)^T = \begin{bmatrix} \hat{\mathbf{i}}_B^A & \hat{\mathbf{j}}_B^A & \hat{\mathbf{k}}_B^A \end{bmatrix}, \quad (61)$$

$$\mathbf{R}_{\text{passive}} = [F_A]_B = ([\mathcal{F}_A]_B)^T = \begin{bmatrix} \hat{\mathbf{i}}_A^B & \hat{\mathbf{j}}_A^B & \hat{\mathbf{k}}_A^B \end{bmatrix}, \quad (62)$$

where in both cases frame F_B is obtained by rotating frame F_A axes. The apparent inverse relationship between the active and passive rotation matrices is not accidental. The active rotation matrix describes what happens to vector components in the fixed original frame F_A from the perspective of the rotated frame F_B :

$$F_B = F_A \mathbf{R}_B^A, \quad (63)$$

and the passive rotation matrix describes what happens to the vector components in the rotated frame F_B from the perspective of the fixed original frame F_A :

$$F_A = F_B \mathbf{R}_A^B. \quad (64)$$

Therefore, we have:

$$\mathbf{x}^A = \underbrace{\mathbf{R}_{\text{active}}}_{\mathbf{R}_B^A} \mathbf{x}^B, \quad \mathbf{x}^B = \underbrace{\mathbf{R}_{\text{passive}}}_{\mathbf{R}_A^B} \mathbf{x}^A, \quad (65)$$

where \mathbf{R}_B^A is called the *rotation matrix from frame F_B to frame F_A* . Since the two have inverse relationship, we have:

$$\mathbf{R}_{\text{active}} = \mathbf{R}_B^A = (\mathbf{R}_A^B)^T = \mathbf{R}_{\text{passive}}^T, \quad (66)$$

where $\mathbf{R}_{\text{active}}, \mathbf{R}_{\text{passive}}, \mathbf{R}_A^B, \mathbf{R}_B^A \in SO(3)$.

3 Geometry and Topology of $SO(3)$

Beyond its algebraic structure, $SO(3)$ has a rich geometric and topological structure that governs the global behavior of rotations. In this section, we equip $SO(3)$ with a topology and establish its key topological properties [Tu11, Men90, Sto23]. These properties place fundamental limitations on rotation representation.

3.1 $SO(3)$ as a Topological Space

$SO(3)$ defined in Section 2 is just a group, i.e., a set equipped with a composition rule. The algebraic structure says nothing about how the elements of $SO(3)$ relate *geometrically*: given two rotations, the group axioms provide no notion of whether they are “close” or “far apart”, what it means for a sequence of rotations to converge, or whether two rotations can be connected by a continuous path. To study these properties on $SO(3)$, we must equip it with a topology.

3.1.1 Topology and Subspace Topology

A *topology* on a set X is a collection \mathcal{T} of subsets of X (called *open sets*) that contains both \emptyset and X , is closed under arbitrary unions, and is closed under finite intersections. The pair (X, \mathcal{T}) is called a *topological space*.

The standard example of a topological space is the Euclidean space \mathbb{R}^n equipped with the *Euclidean topology*. An *open ball* centered at $\mathbf{c} \in \mathbb{R}^n$ with radius $r > 0$ is defined as

$$B(\mathbf{c}, r) := \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{c}\| < r\}. \quad (67)$$

A subset $U \subseteq \mathbb{R}^n$ is *open* if for every $\mathbf{c} \in U$, there exists an $r > 0$ such that $B(\mathbf{c}, r) \subseteq U$. The collection of all such open subsets defines the Euclidean topology on \mathbb{R}^n . Intuitively, the Euclidean topology captures the usual notion of closeness in ordinary Euclidean space: a set is open if, around every point in the set, one can move a little in any direction without leaving the set.

Let (X, \mathcal{T}) be a topological space and let $Y \subseteq X$. The *subspace topology* on Y is defined as:

$$\mathcal{T}_Y := \{U \cap Y \mid U \in \mathcal{T}\}. \quad (68)$$

This makes (Y, \mathcal{T}_Y) a topological space in its own right: a subset of Y is open if and only if it is the intersection of Y with an open set of X .

3.1.2 Topology on $SO(3)$

The matrix representation of $SO(3)$ is defined in Section 2.3 as:

$$SO(3) = \{\mathbf{R} \in \mathbb{R}^{3 \times 3} \mid \mathbf{R}\mathbf{R}^T = \mathbf{I}, \det(\mathbf{R}) = 1\}. \quad (69)$$

Since the space of matrices in $\mathbb{R}^{3 \times 3}$ is isomorphic to \mathbb{R}^9 as a vector space, i.e., $\mathbb{R}^{3 \times 3} \cong \mathbb{R}^9$, and \mathbb{R}^9 carries the Euclidean topology, we equip $SO(3)$ with the subspace topology inherited from \mathbb{R}^9 :

$$\mathcal{T}_{SO(3)} := \{U \cap SO(3) \mid U \text{ open in } \mathbb{R}^9\}. \quad (70)$$

Therefore, $(SO(3), \mathcal{T}_{SO(3)})$ is a topological space. Under this topology, two rotations are considered “nearby” when their corresponding matrix entries are close in the Euclidean sense. Small perturbations to a rotation produce nearby rotations, matching the physical intuition that slightly adjusting an orientation of a frame produces a slightly different orientation.

The group operations of $SO(3)$ (matrix multiplication and inversion) are continuous with respect to this topology. As a result, $SO(3)$ forms a *topological group*: continuous trajectories of rotations remain continuous under composition, and the inverse of a continuously varying rotation varies continuously. This property is essential for attitude estimation, where orientations of rigid bodies evolve smoothly in time.

3.2 Visualizing $SO(3)$

Having defined the topology on $SO(3)$, we now discuss two standard and complementary ways to visualize $SO(3)$: the *ball model*, in which each rotation $r \in SO(3)$ is a point in a closed ball of radius π with *antipodal* boundary points identified, and the *double cover by S^3* , in which each rotation corresponds to a pair of antipodal points on the unit sphere in \mathbb{R}^4 .

3.2.1 Ball Model with Antipodal Boundary Identification

As discussed in Section 2.4, Euler’s rotation theorem implies that every rotation can be described by a rotation vector $\vec{\phi} = \vec{\mathbf{u}}\theta \in \mathbb{V}$ where $\vec{\mathbf{u}} \in \mathbb{V}$ is the unit axis of rotation and $\theta \in [0, \pi]$ is the angle of rotation. The set of all rotation vectors fills a closed ball of radius π in three-dimensional space:

$$B_\pi := \{\vec{\phi} \in \mathbb{V} \mid \|\vec{\phi}\| = \theta \leq \pi\}. \quad (71)$$

In this *ball model*, each element $r\{\vec{\phi}\} \in SO(3)$ is a point in three-dimensional space with:

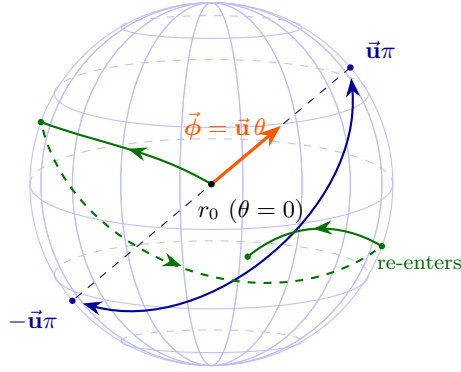


Figure 3: Visualizing $SO(3)$ with a ball model B_π with origin r_0 with antipodal points at $\pm \bar{\mathbf{u}}\pi$. A continuous path r re-enters through its antipodal points (shown in green)

1. Origin or the center of the ball (i.e., $\vec{\phi} = \vec{\mathbf{0}}$) corresponds to the identity rotation ($\theta = 0$).
2. Each point in the interior of the ball corresponds to a unique rotation described by the pair $(\bar{\mathbf{u}}, \theta)$ with $\theta < \pi$.
3. Each point on the boundary of the sphere ($\theta = \pi$) corresponds to rotations by exactly π .
4. For each unique rotation, the axis of rotation specifies which direction the point is from the origin and the angle specifies how far the point is from the center.
5. Small rotations lie near the origin and large rotations lie near the boundary.

The ball model by itself is not a complete picture of $SO(3)$. At the boundary of the ball, a rotation by $\theta = \pi$ about axis $\bar{\mathbf{u}}$ corresponds to the same physical rotation as a rotation by $\theta = \pi$ about axis $-\bar{\mathbf{u}}$:

$$r\{\bar{\mathbf{u}}\pi\} \equiv r\{-\bar{\mathbf{u}}\pi\}. \quad (72)$$

Therefore, every pair of *antipodal* points on the boundary sphere, i.e., points of the form $\bar{\mathbf{u}}\pi$ and $-\bar{\mathbf{u}}\pi$, must be *identified* (i.e., treated as the same point). Denoting this identification by \sim , we have:

$$\bar{\mathbf{u}}\pi \sim -\bar{\mathbf{u}}\pi, \quad \forall \bar{\mathbf{u}} \in \mathbb{V} \text{ with } \|\bar{\mathbf{u}}\| = 1. \quad (73)$$

The resulting quotient space B_π / \sim is *homeomorphic* to the three-dimensional *real projective space* \mathbb{RP}^3 , the space of all lines through the origin in \mathbb{R}^4 . That is, there exists bijective map between B_π / \sim and \mathbb{RP}^3 that is continuous in both directions. Hence, the two spaces have the same topological structure: open sets, neighborhoods, convergence, connectedness, compactness, and continuity are preserved under this correspondence. Therefore:

$$SO(3) \cong B_\pi / \sim \cong \mathbb{RP}^3. \quad (74)$$

This identification has a concrete geometric consequence. A continuous path in $SO(3)$ that reaches the boundary of the ball at some point $\pi\bar{\mathbf{u}}$ re-enters at the antipodal point $-\pi\bar{\mathbf{u}}$, preserving continuity as illustrated in Figure 3. The path does not “leave” the space; it wraps around through the identification. In other words, the rotation can “move” freely in the interior of the ball, but when it hits the boundary, it can move directly to the other side of the ball.

3.2.2 Double Cover by S^3

Another way to visualize $SO(3)$ is to consider the three-dimensional unit sphere defined as:

$$S^3 := \{\underline{q} \in \mathbb{H} \mid \|\underline{q}\| = 1\}, \quad (75)$$

where \underline{q} is a *unit quaternion* in the space of abstract quaternions \mathbb{H} discussed in Section 6. There exists a continuous, surjective map $p : S^3 \rightarrow SO(3)$ that is exactly two-to-one: for every rotation

$r \in SO(3)$, there exist exactly two points $\underline{q}, -\underline{q} \in S^3$ such that $p(\underline{q}) = p(-\underline{q}) = r$. In other words, $SO(3)$ is obtained from S^3 by identifying each point with its antipode:

$$SO(3) \cong S^3 / \{\pm 1\}. \quad (76)$$

The map p is called a *covering map* (a continuous surjection with uniform finite multiplicity—here, exactly two) and S^3 is called a *double cover* of $SO(3)$. The sphere S^3 is *simply connected* (i.e., every loop on S^3 can be continuously contracted to a point).

3.3 Topological Properties of $SO(3)$

Having equipped $SO(3)$ with a topology, we examine three properties: *compactness*, *connectedness*, and *non-simple-connectedness*. Together, these properties distinguish $SO(3)$ from a vector space and impose fundamental constraints on how rotations can be represented.

3.3.1 Compactness

$SO(3)$ is *compact* [Hal15, p. 11]. Geometrically, compactness means that the space of all rotations is finite in extent, unlike translation in Euclidean space \mathbb{R}^3 , which is unbounded. A fundamental consequence of compactness is that every continuous function on $SO(3)$ attains its maximum and minimum. This is important in optimization problems over rotations. In addition, every sequence in $SO(3)$ has a convergent subsequence, so sequences of rotations cannot escape to infinity.

Compactness also has an important implication for parameterizations. Since $SO(3)$ is compact whereas \mathbb{R}^3 is not, the two spaces cannot be homeomorphic. Consequently, no global three-parameter representation of rotations can be both continuous and nonsingular everywhere. This topological mismatch is one reason why every minimal three-parameter parameterization of rotations must exhibit a singularity or discontinuity somewhere, as discussed in Section 3.5.

By the Heine–Borel theorem, a subset of \mathbb{R}^n is compact if and only if it is closed and bounded [Men90, p. 167]. Therefore, to show that $SO(3)$ is compact, we must show that $SO(3)$ is both closed and bounded in \mathbb{R}^9 .

$SO(3)$ is Closed Define the map $f : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3} \times \mathbb{R}$ as:

$$f(\mathbf{R}) := (\mathbf{R}^T \mathbf{R}, \det(\mathbf{R})). \quad (77)$$

Since matrix multiplication, transposition, and the determinant are all polynomial functions of matrix entries, f is continuous. Furthermore:

$$SO(3) = f^{-1}((\mathbf{I}, 1)) = \{\mathbf{R} \in \mathbb{R}^{3 \times 3} \mid \mathbf{R}^T \mathbf{R} = \mathbf{I}, \det(\mathbf{R}) = 1\}. \quad (78)$$

The singleton $\{(\mathbf{I}, 1)\}$ is a closed set in $\mathbb{R}^{3 \times 3} \times \mathbb{R}$, and the preimage of a closed set under a continuous map is closed. Therefore, $SO(3)$ is closed in \mathbb{R}^9 .

$SO(3)$ is Bounded For any $\mathbf{R} \in SO(3)$, the orthogonality constraint $\mathbf{R}^T \mathbf{R} = \mathbf{I}$ implies that each column of \mathbf{R} is a unit vector in \mathbb{R}^3 . Therefore, the entries of \mathbf{R} satisfy $|R_{ij}| \leq 1, \forall i, j$. The Frobenius norm can be computed as:

$$\|\mathbf{R}\|_F = \sqrt{\text{tr}(\mathbf{R}^T \mathbf{R})} = \sqrt{\text{tr}(\mathbf{I})} = \sqrt{3}. \quad (79)$$

Therefore, every element of $SO(3)$ has the same Frobenius norm $\sqrt{3}$, so $SO(3)$ lies on a sphere of radius $\sqrt{3}$ in \mathbb{R}^9 and is bounded.

3.3.2 Connectedness

$SO(3)$ is *path-connected* [Hal15, p. 12]. Geometrically, path-connectedness means that every rotation can be reached from every other rotation by a continuous rotational motion. There are no disconnected “gaps” or isolated elements in $SO(3)$. In particular, between any two rotations there exists a continuous trajectory in $SO(3)$, which underlies interpolation methods such as spherical linear interpolation (SLERP).

$SO(3)$ is Path-Connected A topological space (X, \mathcal{T}) is path connected if any two points in X can be joined by a continuous path lying entirely within X [Men90, p. 134]. To show that $SO(3)$ is path-connected, we show that every rotation $r \in SO(3)$ can be connected to the identity r_0 by a continuous path in $SO(3)$. Let $\bar{\mathbf{u}} \in \mathbb{V}$ be a unit axis of rotation and $\theta \in (0, \pi]$ be an angle of rotation. Using axis-angle pair as defined in Section 2.4, define a path:

$$\gamma : [0, 1] \rightarrow SO(3), \quad \gamma(t) := r\{(t\theta) \bar{\mathbf{u}}\}, \quad (80)$$

which continuously varies the rotation angle from 0 to θ while keeping the axis of rotation fixed. At $t = 0$, $\gamma(0) = r\{0\bar{\mathbf{u}}\} = r_0$, and at $t = 1$, $\gamma(1) = r\{\theta\bar{\mathbf{u}}\} = r$. Since $\gamma(t) \in SO(3)$ for all $t \in [0, 1]$ and γ is continuous, the path connects r to r_0 .

Since every element of $SO(3)$ can be connected to the identity by a continuous path, any two elements $r_1, r_2 \in SO(3)$ can be connected to each other (by concatenating the path from r_1 to r_0 with the path from r_0 to r_2). Therefore, $SO(3)$ is path-connected.

$SO(3)$ is Connected A topological space (X, \mathcal{T}) is *connected* if it cannot be written as a union of two disjoint non-empty open sets. Equivalently, a topological space is connected if and only if it is path-connected. Hence, $SO(3)$ is connected [Hal15, p. 12].

3.3.3 Non-Simply-Connectedness

$SO(3)$ is *non-simply-connected* [Hal15, p. 15–16]. Geometrically, it means that a rotation of 360° is not topologically equivalent to a no rotation, but a 720° rotation is. This can be illustrated by *Dirac’s belt trick*. If one end of a belt is fixed and the other end is attached to a rotating object, then after a 360° rotation, the belt becomes twisted in a way that cannot be continuously removed without further rotating the object. In contrast, after a 720° rotation, the belt can be continuously untwisted without changing the final orientation of the object.

$SO(3)$ is Non-Simply-Connected A closed loop in a topological space (X, \mathcal{T}) based at a point $x_0 \in X$ is a continuous map:

$$\gamma : [0, 1] \rightarrow X, \quad \gamma(0) = \gamma(1) = x_0. \quad (81)$$

A connected topological space (X, \mathcal{T}) is *simply connected* if every closed loop in (X, \mathcal{T}) can be continuously deformed (or contracted) to a single point. If there exist loops that cannot be contracted, (X, \mathcal{T}) is *non-simply-connected*. $SO(3)$ is *non-simply-connected*.

The ball model and the antipodal identification on the boundary defined in Section 3.2.1 gives rise to two topologically distinct types of closed loops in $SO(3)$ called *contractible* and *non-contractible* loops:

1. Consider a closed loop that stays entirely within the interior of the ball B_π , never touching the boundary. Such a loop can be continuously contracted to a point, just as any loop in \mathbb{R}^3 can. These loops are *contractible*.
2. Consider a path that starts at the origin, travels to the boundary point $\bar{\mathbf{u}}\pi$, and through the antipodal identification, re-enters at $-\bar{\mathbf{u}}\pi$, then returns to the origin. This path is a closed loop in $SO(3)$ since it starts and ends at the identity. It is called *non-contractible loop* and cannot be continuously deformed to a point.

Therefore, $SO(3)$ is non-simply-connected due to the existence of non-contractible loops.

For any path-connected topological space, one can define an object called the *fundamental group*; a group of the equivalence classes under *homotopy* (i.e., continuous interpolation between two loops) of the loops contained in the space. It records information about the basic shape, or holes, of the topological space. The distinction between contractible and non-contractible loops is made precise by the fundamental group $\pi_1(X)$ of a topological space (X, \mathcal{T}) . Two loops are considered equivalent if one can be continuously deformed into the other. For $SO(3)$, the fundamental group is:

$$\pi_1(SO(3)) \cong \mathbb{Z}_2 = \{0, 1\}, \quad (82)$$

where \mathbb{Z}_2 is the cyclic group of order two. This means there are exactly two equivalence classes of closed loops in $SO(3)$:

1. Loops that can be contracted to a point (class 0).
2. Loops that cannot be contracted to a point, but whose double traversal is contractible (class 1).

This result can be understood through the double cover $S^3 \rightarrow SO(3)$ from Section 3.2.2. Since S^3 is simply connected ($\pi_1(S^3) = 0$), the fundamental group of $SO(3)$ is determined by the covering: $\pi_1(SO(3)) \cong \mathbb{Z}_2$, reflecting the two-to-one nature of the covering map.

3.4 $SO(3)$ as a Smooth Manifold

A *manifold* of dimension n is a topological space that locally resembles Euclidean space \mathbb{R}^n where every point has a neighborhood that is homeomorphic to an open subset of \mathbb{R}^n . A *smooth manifold* is a manifold on which transitions between such local descriptions are infinitely differentiable, so that derivatives, tangent vectors, and Taylor expansions are well-defined intrinsically. Examples include curves ($n = 1$), the sphere S^2 ($n = 2$), and the rotation group $SO(3)$ ($n = 3$).

Since $SO(3)$ is a three-dimensional smooth manifold, every rotation has a neighborhood that can be smoothly parameterized by three real numbers. The three-parameter representations (i.e., rotation vectors and Euler angles) are such local parameterizations called *local charts* on $SO(3)$. The smooth structure makes it possible to differentiate curves of rotations and to define a tangent space at each point. The tangent space at the identity carries additional algebraic structure that links rotations to angular velocity as discussed Section 10.1.

3.5 Consequences of $SO(3)$ Representations

The topological properties established in Section 3.3 impose fundamental constraints on how rotations can be parameterized. These constraints are not artifacts of any particular representation; they are intrinsic to the topology of $SO(3)$ itself.

The group $SO(3)$ is a three-dimensional smooth manifold (Section 3.4), so any minimal representation uses three independent parameters locally. A representation of $SO(3)$ is said to be *globally valid* if it provides a *diffeomorphism* (a smooth bijection with a smooth inverse) between a parameter space \mathcal{P} and $SO(3)$, without singularities, discontinuities, or ambiguities over its entire domain. If this fails, the representation is only *locally valid*. A fundamental consequence of the topology of $SO(3)$ is that no globally valid representation exists with parameter space \mathbb{R}^3 .

3.5.1 Three-Parameter Representations

Three-parameter representations include *Euler angles* and rotation vectors (axis-angle). $SO(3)$ cannot be globally parameterized by three real numbers i.e., there does not exist a homeomorphism $f: \mathbb{R}^3 \rightarrow SO(3)$ (or from any open subset of \mathbb{R}^3 onto all of $SO(3)$), for two independent topological reasons:

1. *Compactness obstruction:* $SO(3)$ is compact (Section 3.3.1), while \mathbb{R}^3 is not. Since homeomorphisms preserve compactness, no homeomorphism between them can exist.
2. *Fundamental group obstruction:* $SO(3)$ has fundamental group $\pi_1(SO(3)) \cong \mathbb{Z}_2$ as defined in (82), while \mathbb{R}^3 is simply connected ($\pi_1(\mathbb{R}^3) = 0$). Since the fundamental group is a *topological invariant* (a property preserved by homeomorphisms), $SO(3)$ and \mathbb{R}^3 are not homeomorphic.

Consequently, no single three-parameter representation can be globally valid. Any such representation must fail globally in at least one way, for example through:

1. *Singularities:* Points at which the parameterization becomes degenerate (e.g., gimbal lock in Euler angles).
2. *Discontinuities:* Points at which the parameterization is not continuous (e.g., the jump at $\theta = \pi$ in rotation vectors).
3. *Non-injectivity:* Distinct parameter values representing the same rotation.
4. *Non-surjectivity:* Failure to cover all of $SO(3)$.

3.5.2 Four-Parameter Representations

The double cover $S^3 \rightarrow SO(3)$ (Section 3.2.2) represents rotations using four parameters subject to one constraint. The unit quaternion representation does exactly this: it uses four real numbers $\mathbf{q} \in \mathbb{R}^4$ constrained by $\|\mathbf{q}\| = 1$. This gives a global, continuous, and non-singular representation of rotations. The cost is a two-to-one ambiguity: \mathbf{q} and $-\mathbf{q}$ represent the same rotation. This sign ambiguity reflects the fact that $\pi_1(SO(3)) \cong \mathbb{Z}_2$.

3.5.3 Nine-Parameter Representations

Rotation matrices in $\mathbb{R}^{3 \times 3}$ use nine entries constrained by the orthonormality condition $\mathbf{R}^T \mathbf{R} = \mathbf{I}$, leaving three degrees of freedom, with the additional requirement $\det(\mathbf{R}) = 1$. This representation is globally valid and non-singular, at the cost of redundancy.

III Rotation Representations

In this part, we discuss the representations of $SO(3)$ and the transformation between them: *rotation matrices*, the *axis-angle* representation, the *unit quaternion* representation, and *Euler angles*. Although all of these represent the same underlying rotation group, they are not equivalent as parameterizations: they differ in global validity, uniqueness, redundancy, and the presence of singularities, as discussed in Section 3.5.

4 Rotation Matrices

As defined in Section 2.3, rotation matrices constitute the matrix representations of the rotation group $SO(3)$:

$$SO(3) = \{ \mathbf{R} \in \mathbb{R}^{3 \times 3} \mid \mathbf{R}\mathbf{R}^T = \mathbf{I}, \det(\mathbf{R}) = 1 \}. \quad (83)$$

It is a globally valid representation as discussed in Section 3.5.

4.1 Notation and Terminology of Rotation Matrices

Some authors may refer rotation matrices as passive rotation matrices when the frames are explicit (\mathbf{R}_B^A or \mathbf{R}_A^B). In this convention, rotation matrices are called active rotation matrices only when they are associated with physical rotation of vectors in a single reference frame.

In this document, we denote rotation matrices simply as \mathbf{R} if it describes an active rotation operation with a single reference frame. Otherwise, we denote rotation matrices as $\mathbf{R}_{\text{src}}^{\text{dest}}$ to make the frames axes relationship explicit, i.e., rotation matrix from a *source* frame to a *destination* frame.

Rotation matrices are also commonly referred to as *direction cosine matrices* (DCM), *orientations*, and *coordinate transformation matrices*.

4.1.1 Direction Cosine Matrix

Consider a rotation matrix from frame F_A to frame F_B . From (59), we have:

$$\mathbf{R}_A^B = [F_A]_B \quad (84)$$

$$= \begin{bmatrix} \hat{\mathbf{i}}_B \cdot \hat{\mathbf{i}}_A & \hat{\mathbf{i}}_B \cdot \hat{\mathbf{j}}_A & \hat{\mathbf{i}}_B \cdot \hat{\mathbf{k}}_A \\ \hat{\mathbf{j}}_B \cdot \hat{\mathbf{i}}_A & \hat{\mathbf{j}}_B \cdot \hat{\mathbf{j}}_A & \hat{\mathbf{j}}_B \cdot \hat{\mathbf{k}}_A \\ \hat{\mathbf{k}}_B \cdot \hat{\mathbf{i}}_A & \hat{\mathbf{k}}_B \cdot \hat{\mathbf{j}}_A & \hat{\mathbf{k}}_B \cdot \hat{\mathbf{k}}_A \end{bmatrix} \in SO(3). \quad (85)$$

Since its elements are the dot products between orthonormal basis vectors of two frames, it can be described by the cosines of the angles between the basis vectors. Hence, rotation matrices are also called *direction cosine matrices*.

4.1.2 Orientation Matrix

Rotation matrices are also called *orientation matrices* since they describe the basis vector orientations of one frame relative to another frame. The rotation matrix from frame F_A to frame F_B denoted as \mathbf{R}_A^B describes the *orientation of frame F_A relative to frame F_B* since it contains the information about how the axes of F_A are oriented as seen from F_B :

$$F_A = F_B \mathbf{R}_A^B. \quad (86)$$

Conversely, $\mathbf{R}_B^A = [F_B]_A$ describes the orientation of frame F_B relative to frame F_A .

In state estimation, orientation matrix describes *attitude* of a rigid body. Let F_B be a body-fixed frame attached to the center of mass of a rigid body. Let F_G be a global reference frame. Then, the attitude of the body can be described by the rotation matrix \mathbf{R}_B^G .

4.1.3 Coordinate Transformation Matrix

In state estimation, kinematic properties, such as position, are commonly transformed from one frame to another frame. Let F_A and F_B be frame and let $\mathbf{p}^A, \mathbf{p}^B \in \mathbb{R}^3$ be the component position vectors in F_A and F_B , respectively. We have:

$$\mathbf{p}^B = \mathbf{R}_A^B \mathbf{p}^A, \quad (87)$$

Some authors use the notation $\mathbf{C}_A^B := \mathbf{R}_A^B$ which is called the *coordinate transformation matrices*. The distinction between the coordinate transformation matrix and the rotation matrix is purely terminological.

4.2 Rotation Matrix Properties

Let F_A and F_B be frames. Let $\mathbf{R}_A^B \in SO(3)$ be the rotation matrix from frame F_A to frame F_B . From Section 2.5.3, we have:

$$\begin{aligned} \mathbf{R}_A^B &= [F_A]_B = ([\mathcal{F}_A]_B) \\ &= F_B^T F_A = \mathcal{F}_B \mathcal{F}_A^T, \\ \mathbf{R}_B^A &= [F_B]_A = ([\mathcal{F}_B]_A) \\ &= F_A^T F_B = \mathcal{F}_A \mathcal{F}_B^T. \end{aligned} \quad (88)$$

Since \mathbf{R}_A^B and \mathbf{R}_B^A are inverse rotations, we have:

$$\boxed{\mathbf{R}_A^B = (\mathbf{R}_B^A)^{-1} = (\mathbf{R}_B^A)^T}. \quad (89)$$

Equation (88) can be rewritten as:

$$\boxed{\begin{aligned} F_A &= F_B \mathbf{R}_A^B, & \mathcal{F}_A &= \mathbf{R}_B^A \mathcal{F}_B, \\ F_B &= F_A \mathbf{R}_B^A, & \mathcal{F}_B &= \mathbf{R}_A^B \mathcal{F}_A. \end{aligned}} \quad (90)$$

4.3 Rotation Matrix Composition

Let F_C be a frame. The rotation matrices can be composed such that:

$$\mathbf{R}_C^A = \mathbf{R}_B^A \mathbf{R}_C^B. \quad (91)$$

To see this, let $\mathbf{x}^A, \mathbf{x}^B, \mathbf{x}^C \in \mathbb{R}^3$ be component vectors of $\vec{\mathbf{x}} \in \mathbb{V}$ in F_A, F_B , and F_C , respectively. We have:

$$\begin{aligned} \mathbf{x}^A &= \mathbf{R}_B^A \mathbf{x}^B \\ &= \mathbf{R}_B^A (\mathbf{R}_C^B \mathbf{x}^C) \\ &= \underbrace{(\mathbf{R}_B^A \mathbf{R}_C^B)}_{\mathbf{R}_C^A} \mathbf{x}^C. \end{aligned} \quad (92)$$

4.4 Principal Rotation Matrices

A *principal rotation* is a rotation about a single axis of a frame. Let F_A be a frame. For three-dimensional rotations, there are three principal rotation matrices:

$$\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3 \in SO(3), \quad (93)$$

where each correspond to rotation about single axis of rotation: 1-axis (\hat{i}_A), 2-axis (\hat{j}_A), and 3-axis (\hat{k}_A) using the right-hand rule.

Principal rotation matrices can be obtained in two ways: active rotation of a physical vector through an angle θ about the 1-axis, 2-axis, and 3-axis using the right-hand rule using the active rotation operation or a change of basis operation by a passive rotation where frame F_B is obtained by rotating F_A through an angle θ about the 1-axis, 2-axis, and 3-axis of F_A . Table 2 summarizes the principal rotation matrices.

Rotation Type	Principal Rotation Matrix	Frame Relationship
3-axis, angle θ	$\mathbf{R}_3(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{aligned} \mathbf{R}_B^A &= \mathbf{R}_3(\theta), \\ \mathbf{R}_A^B &= \mathbf{R}_3^T(\theta) \end{aligned}$
2-axis, angle θ	$\mathbf{R}_2(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$	$\begin{aligned} \mathbf{R}_B^A &= \mathbf{R}_2(\theta), \\ \mathbf{R}_A^B &= \mathbf{R}_2^T(\theta) \end{aligned}$
1-axis, angle θ	$\mathbf{R}_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$	$\begin{aligned} \mathbf{R}_B^A &= \mathbf{R}_1(\theta), \\ \mathbf{R}_A^B &= \mathbf{R}_1^T(\theta) \end{aligned}$

Table 2: Principal Rotation Matrices: Frame F_B is obtained by rotating frame F_A through an angle θ about the 1-axis, 2-axis, and 3-axis of F_A using the right-hand rule

Derivations of principal rotation matrices about the 3-axis using the two approaches are discussed below. Both derivations are presented for clarity.

4.4.1 3-Axis Principal Rotation Matrix: Active Rotation Approach

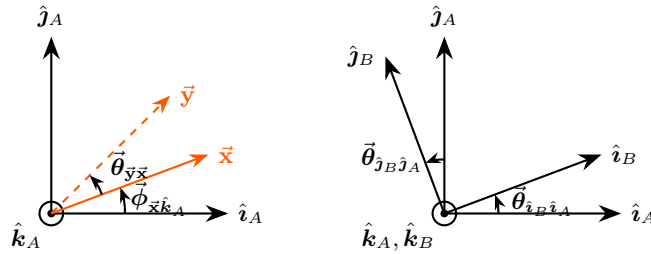


Figure 4: 3-Axis Principal Rotations: (left) Rotation of component vector \mathbf{x}^A by an angle θ about \hat{k}_A . (right) Frame F_A rotation by an angle θ_{BA} about \hat{k}_A to get frame F_B .

Let F_A be a frame and $\vec{\mathbf{x}} \in \mathbb{V}$ a physical vector. The active rotation operation about $\hat{\mathbf{k}}_A$ produces a rotated vector $\vec{\mathbf{y}} \in \mathbb{V}$ as illustrated in Figure 4 (left). The physical angle vector of $\vec{\mathbf{y}}$ relative to $\vec{\mathbf{x}}$ is defined as:

$$\vec{\theta}_{\vec{\mathbf{y}}\vec{\mathbf{x}}} = \theta \hat{\theta}_{\vec{\mathbf{y}}\vec{\mathbf{x}}} = \theta \hat{\mathbf{k}}_A \in \mathbb{V}, \quad (94)$$

where $\theta \in \mathbb{R}$ is the angle of rotation. The components of $\vec{\mathbf{y}}$ in frame F_A can be obtained as:

$$y_1 = \hat{\mathbf{i}}_A \cdot \vec{\mathbf{y}}, \quad y_2 = \hat{\mathbf{j}}_A \cdot \vec{\mathbf{y}}, \quad y_3 = \hat{\mathbf{k}}_A \cdot \vec{\mathbf{y}}. \quad (95)$$

Let $\vec{\phi}_{\vec{\mathbf{x}}\hat{\mathbf{i}}_A} = \phi \hat{\phi}_{\vec{\mathbf{x}}\hat{\mathbf{i}}_A}$ be the angle vector of $\vec{\mathbf{x}}$ relative to $\hat{\mathbf{i}}_A$. The components of physical vector $\vec{\mathbf{x}}$ in F_A can be obtained as:

$$x_1 = \hat{\mathbf{i}}_A \cdot \vec{\mathbf{x}} = \|\vec{\mathbf{x}}\| \cos \phi, \quad (96)$$

$$x_2 = \hat{\mathbf{j}}_A \cdot \vec{\mathbf{x}} = \|\vec{\mathbf{x}}\| \sin \phi, \quad (97)$$

$$x_3 = \hat{\mathbf{k}}_A \cdot \vec{\mathbf{x}}. \quad (98)$$

Since $\vec{\mathbf{x}}$ is rotated by θ , the angle vector of $\vec{\mathbf{y}}$ relative to $\hat{\mathbf{i}}_A$ is $\vec{\theta}_{\vec{\mathbf{y}}\vec{\mathbf{x}}} + \vec{\phi}_{\vec{\mathbf{x}}\hat{\mathbf{i}}_A}$, and therefore:

$$y_1 = \|\vec{\mathbf{x}}\| \cos(\theta + \phi), \quad (99)$$

$$y_2 = \|\vec{\mathbf{x}}\| \sin(\theta + \phi), \quad (100)$$

$$y_3 = x_3. \quad (101)$$

Using angle addition identities and $\cos \phi = x_1/\|\vec{\mathbf{x}}\|$, $\sin \phi = x_2/\|\vec{\mathbf{x}}\|$:

$$y_1 = \|\vec{\mathbf{x}}\| \left(\frac{x_1}{\|\vec{\mathbf{x}}\|} \cos \theta - \frac{x_2}{\|\vec{\mathbf{x}}\|} \sin \theta \right) = x_1 \cos \theta - x_2 \sin \theta, \quad (102)$$

$$y_2 = \|\vec{\mathbf{x}}\| \left(\frac{x_2}{\|\vec{\mathbf{x}}\|} \cos \theta + \frac{x_1}{\|\vec{\mathbf{x}}\|} \sin \theta \right) = x_1 \sin \theta + x_2 \cos \theta, \quad (103)$$

$$y_3 = x_3, \quad (104)$$

or in matrix form:

$$\mathbf{y}^A = \mathbf{R}_3(\theta) \mathbf{x}^A, \quad (105)$$

where:

$$\mathbf{R}_3(\theta) := \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (106)$$

is the the 3-axis principle rotation matrix.

4.4.2 3-Axis Principal Rotation Matrix: Passive Rotation Approach

Frame F_B is obtained by rotating the basis vectors of frame F_A as illustrated in Figure 4 (right). The rotation is about the 3-axis ($\hat{\mathbf{k}}_A$) of frame F_A by an angle $\theta := \theta_{\hat{\mathbf{i}}_B \hat{\mathbf{i}}_A} \equiv \theta_{\hat{\mathbf{j}}_B \hat{\mathbf{j}}_A} \in \mathbb{R}$. The physical angle vector of $\hat{\mathbf{i}}_B$ relative to $\hat{\mathbf{i}}_A$ (or equivalently, $\hat{\mathbf{j}}_B$ relative to $\hat{\mathbf{j}}_A$) is defined as:

$$\vec{\theta}_{\hat{\mathbf{i}}_B \hat{\mathbf{i}}_A} = \theta \hat{\theta}_{\hat{\mathbf{i}}_B \hat{\mathbf{i}}_A} = \theta \hat{\mathbf{k}}_A. \quad (107)$$

We define the angle vector $\vec{\theta}_{BA} \in \mathbb{V}$ to denote the relative angle vector between frame F_B and frame F_A . Hence, we have:

$$\vec{\theta}_{BA} = \theta \hat{\theta}_{BA} = \theta \hat{\mathbf{k}}_A. \quad (108)$$

The basis vectors of F_B can be decomposed in F_A as:

$$\hat{\mathbf{i}}_B = \cos \theta \hat{\mathbf{i}}_A + \sin \theta \hat{\mathbf{j}}_A, \quad (109)$$

$$\hat{\mathbf{j}}_B = -\sin \theta \hat{\mathbf{i}}_A + \cos \theta \hat{\mathbf{j}}_A, \quad (110)$$

$$\hat{\mathbf{k}}_B = \hat{\mathbf{k}}_A. \quad (111)$$

Rewriting in column vectrix form using (5):

$$\mathcal{F}_B = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathcal{F}_A, \quad (112)$$

or equivalently:

$$F_B = F_A \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (113)$$

Hence, we have:

$$\mathbf{R}_3(\theta) := \mathbf{R}_B^A = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (114)$$

5 Axis-Angle Representation

As discussed in Section 2.4, Euler's rotation theorem implies rotation can be represented by a unique *axis-angle* pair in three-dimensional space. The *angle of rotation* is denoted as $\theta \in [0, \pi]$ and the *unit axis of rotation* is denoted as $\vec{\mathbf{u}} \in \mathbb{V}$. The *rotation vector* is defined as:

$$\vec{\phi} := \theta \vec{\mathbf{u}} \in \mathbb{V}. \quad (115)$$

The rotation operation $r\{\vec{\phi}\}$ rotates a physical vector by the angle θ following the right-hand rule along the axis of rotation $\vec{\mathbf{u}}$.

Rotation vectors suffer from discontinuity at $\theta = \pi$, where the rotations of angle π about the axes $\pm \vec{\mathbf{u}}$ represent the same rotation as discussed in Section 3.2.1. Additionally, the representation becomes singular at $\theta = 0$. Nonetheless, axis-angle representation is crucial in state estimation as it enables rotations to be linearly interpolated.

Let F_A and F_B be frames with basis vectors $\{\hat{\mathbf{i}}_A, \hat{\mathbf{j}}_A, \hat{\mathbf{k}}_A\}$ and $\{\hat{\mathbf{i}}_B, \hat{\mathbf{j}}_B, \hat{\mathbf{k}}_B\}$, respectively. Let $\vec{\phi}_{BA} = \theta \mathbf{u} \in \mathbb{V}$ be the *rotation vector of F_B relative to F_A* . The rotation operation $r\{\vec{\phi}_{BA}\}$ describes the same relative orientation in the following equivalent ways:

1. It is the rotation that maps the basis vectors of F_A to the basis vectors of F_B .
2. It is the rotation that carries frame F_A into frame F_B .
3. It encodes the orientation of frame F_B relative to frame F_A .

5.1 Axis-Angle Representation to Rotation Matrix

The component vector of the rotation vector $\vec{\phi}_{BA}$ resolved in F_A and F_B are defined as:

$$\left[\vec{\phi}_{BA}\right]_A = \phi_{BA}^A = \mathbf{u}^A \theta, \quad \left[\vec{\phi}_{BA}\right]_B = \phi_{BA}^B = \mathbf{u}^B \theta. \quad (116)$$

However, the resolving frame does not matter (discussed in 5.3) and therefore we define:

$$\begin{aligned} \phi_{BA}^{B/A} &\equiv \phi_{BA}^{A/B} := \phi_{BA}^A \equiv \phi_{BA}^B \in \mathbb{R}^3, \\ \mathbf{u}^{B/A} &\equiv \mathbf{u}^{A/B} := \mathbf{u}^A \equiv \mathbf{u}^B \in \mathbb{R}^3. \end{aligned} \quad (117)$$

The corresponding rotation matrix parameterized by $\phi_{BA}^{B/A}$ is defined as:

$$\mathbf{R} := \mathbf{R}\{\phi_{BA}^{B/A}\} \in SO(3). \quad (118)$$

To find the relationship between rotation vectors and rotation matrices, we can obtain frame F_B by rotating frame F_A by the rotation vector $\vec{\phi}_{BA} = \theta \vec{\mathbf{u}}$ using *Rodrigues'* rotation formula (46):

$$\begin{aligned} \hat{\mathbf{i}}_B &= \hat{\mathbf{i}}_A \cos \theta + (\vec{\mathbf{u}} \times \hat{\mathbf{i}}_A) \sin \theta + \vec{\mathbf{u}} (\vec{\mathbf{u}} \cdot \hat{\mathbf{i}}_A) (1 - \cos \theta), \\ \hat{\mathbf{j}}_B &= \hat{\mathbf{j}}_A \cos \theta + (\vec{\mathbf{u}} \times \hat{\mathbf{j}}_A) \sin \theta + \vec{\mathbf{u}} (\vec{\mathbf{u}} \cdot \hat{\mathbf{j}}_A) (1 - \cos \theta), \\ \hat{\mathbf{k}}_B &= \hat{\mathbf{k}}_A \cos \theta + (\vec{\mathbf{u}} \times \hat{\mathbf{k}}_A) \sin \theta + \vec{\mathbf{u}} (\vec{\mathbf{u}} \cdot \hat{\mathbf{k}}_A) (1 - \cos \theta). \end{aligned} \quad (119)$$

Let the component vector of $\vec{\mathbf{u}}$ in frame F_A be $\mathbf{u}^A = [u_1, u_2, u_3]$ such that:

$$\begin{aligned} \vec{\mathbf{u}} &= F_A \mathbf{u}^A \\ &= u_1 \hat{\mathbf{i}}_A + u_2 \hat{\mathbf{j}}_A + u_3 \hat{\mathbf{k}}_A. \end{aligned} \quad (120)$$

From (120), we have:

$$\begin{aligned} \vec{\mathbf{u}} \times \hat{\mathbf{i}}_A &= (u_1 \hat{\mathbf{i}}_A + u_2 \hat{\mathbf{j}}_A + u_3 \hat{\mathbf{k}}_A) \times \hat{\mathbf{i}}_A = -u_2 \hat{\mathbf{k}}_A + u_3 \hat{\mathbf{j}}_A, & \vec{\mathbf{u}} \cdot \hat{\mathbf{i}}_A &= u_1, \\ \vec{\mathbf{u}} \times \hat{\mathbf{j}}_A &= (u_1 \hat{\mathbf{i}}_A + u_2 \hat{\mathbf{j}}_A + u_3 \hat{\mathbf{k}}_A) \times \hat{\mathbf{j}}_A = u_1 \hat{\mathbf{k}}_A - u_3 \hat{\mathbf{i}}_A, & \vec{\mathbf{u}} \cdot \hat{\mathbf{j}}_A &= u_2, \\ \vec{\mathbf{u}} \times \hat{\mathbf{k}}_A &= (u_1 \hat{\mathbf{i}}_A + u_2 \hat{\mathbf{j}}_A + u_3 \hat{\mathbf{k}}_A) \times \hat{\mathbf{k}}_A = -u_1 \hat{\mathbf{j}}_A + u_2 \hat{\mathbf{i}}_A, & \vec{\mathbf{u}} \cdot \hat{\mathbf{k}}_A &= u_3, \end{aligned} \quad (121)$$

Substituting above into (119) yields to:

$$\begin{aligned} \hat{\mathbf{i}}_B &= (\cos \theta + u_1^2 (1 - \cos \theta)) \hat{\mathbf{i}}_A \\ &\quad + (u_3 \sin \theta + u_1 u_2 (1 - \cos \theta)) \hat{\mathbf{j}}_A \\ &\quad + (-u_2 \sin \theta + u_1 u_3 (1 - \cos \theta)) \hat{\mathbf{k}}_A, \\ \hat{\mathbf{j}}_B &= (-u_3 \sin \theta + u_1 u_2 (1 - \cos \theta)) \hat{\mathbf{i}}_A \\ &\quad + (\cos \theta + u_2^2 (1 - \cos \theta)) \hat{\mathbf{j}}_A \\ &\quad + (u_1 \sin \theta + u_2 u_3 (1 - \cos \theta)) \hat{\mathbf{k}}_A, \\ \hat{\mathbf{k}}_B &= (u_2 \sin \theta + u_1 u_3 (1 - \cos \theta)) \hat{\mathbf{i}}_A \\ &\quad + (-u_1 \sin \theta + u_2 u_3 (1 - \cos \theta)) \hat{\mathbf{j}}_A \\ &\quad + (\cos \theta + u_3^2 (1 - \cos \theta)) \hat{\mathbf{k}}_A. \end{aligned} \quad (122)$$

Rewriting the above in matrix form yields:

$$\mathcal{F}_B = \begin{bmatrix} \cos \theta + u_1^2 (1 - \cos \theta) & u_1 u_2 (1 - \cos \theta) + u_3 \sin \theta & u_1 u_3 (1 - \cos \theta) - u_2 \sin \theta \\ u_1 u_2 (1 - \cos \theta) - u_3 \sin \theta & \cos \theta + u_2^2 (1 - \cos \theta) & u_2 u_3 (1 - \cos \theta) + u_1 \sin \theta \\ u_1 u_3 (1 - \cos \theta) + u_2 \sin \theta & u_2 u_3 (1 - \cos \theta) - u_1 \sin \theta & \cos \theta + u_3^2 (1 - \cos \theta) \end{bmatrix} \mathcal{F}_A, \quad (123)$$

or equivalently:

$$F_B = F_A \begin{bmatrix} \cos \theta + u_1^2(1 - \cos \theta) & u_1 u_2(1 - \cos \theta) - u_3 \sin \theta & u_1 u_3(1 - \cos \theta) + u_2 \sin \theta \\ u_1 u_2(1 - \cos \theta) + u_3 \sin \theta & \cos \theta + u_2^2(1 - \cos \theta) & u_2 u_3(1 - \cos \theta) - u_1 \sin \theta \\ u_1 u_3(1 - \cos \theta) - u_2 \sin \theta & u_2 u_3(1 - \cos \theta) + u_1 \sin \theta & \cos \theta + u_3^2(1 - \cos \theta) \end{bmatrix}. \quad (124)$$

Hence, from (90), we have:

$$\mathbf{R}_B^A := \begin{bmatrix} \cos \theta + u_1^2(1 - \cos \theta) & u_1 u_2(1 - \cos \theta) - u_3 \sin \theta & u_1 u_3(1 - \cos \theta) + u_2 \sin \theta \\ u_1 u_2(1 - \cos \theta) + u_3 \sin \theta & \cos \theta + u_2^2(1 - \cos \theta) & u_2 u_3(1 - \cos \theta) - u_1 \sin \theta \\ u_1 u_3(1 - \cos \theta) - u_2 \sin \theta & u_2 u_3(1 - \cos \theta) + u_1 \sin \theta & \cos \theta + u_3^2(1 - \cos \theta) \end{bmatrix}. \quad (125)$$

The above matrix can be decomposed as:

$$\begin{aligned} \mathbf{R}_B^A &= \cos \theta \mathbf{I} + (1 - \cos \theta) \begin{bmatrix} u_1^2 & u_1 u_2 & u_1 u_3 \\ u_1 u_2 & u_2^2 & u_2 u_3 \\ u_1 u_3 & u_2 u_3 & u_3^2 \end{bmatrix} + \sin \theta \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \\ &= \cos \theta \mathbf{I} + (1 - \cos \theta) \mathbf{u}^A (\mathbf{u}^A)^T + \sin \theta [\mathbf{u}^A]_{\times}. \end{aligned} \quad (126)$$

Hence, we get:

$$\boxed{\mathbf{R}_B^A = \mathbf{R} \{ \phi_{BA}^{B/A} \} = \cos \theta \mathbf{I} + (1 - \cos \theta) \mathbf{u}^{B/A} (\mathbf{u}^{B/A})^T + \sin \theta [\mathbf{u}^{B/A}]_{\times}}. \quad (127)$$

Equation (127) is called the *matrix form* of the *Rodrigues'* rotation formula.

The inverse rotation can be obtained by the rotation vector $\vec{\phi}_{AB} = -\vec{\phi}_{BA} = -\mathbf{u}\theta \in \mathbb{V}$ with $\phi_{AB}^{A/B} = -\phi_{BA}^{A/B} \in \mathbb{R}^3$:

$$\boxed{\mathbf{R}_A^B = \mathbf{R} \{ \phi_{AB}^{A/B} \} = \cos \theta \mathbf{I} + (1 - \cos \theta) \mathbf{u}^{A/B} (\mathbf{u}^{A/B})^T - \sin \theta [\mathbf{u}^{A/B}]_{\times}}. \quad (128)$$

5.1.1 Exponential Map

Let F_A be and F_B be frames and $\vec{\phi}_{BA} = \mathbf{u}\theta \in \mathbb{V}$ be a rotation vector that encodes the orientation of frame F_B relative to frame F_A . Let $\phi_{BA}^{B/A} \in \mathbb{R}^3$ be the component vector in frame F_A and frame F_B . We have:

$$\phi_{BA}^{B/A} = \theta \mathbf{u}^{B/A} \in \mathbb{R}^3, \quad \mathbf{u}^{B/A} \in \mathbb{S}^2, \quad \theta \in \mathbb{R}, \quad (129)$$

where $\mathbb{S}^2 := \{ \mathbf{v} \in \mathbb{R}^3 \mid \|\mathbf{v}\| = 1 \}$ is the *unit 2-sphere*. Now, consider the matrix exponential of the

skew-symmetric matrix $\left[\phi_{BA}^{B/A}\right]_{\times}$:

$$\begin{aligned}
\exp\left(\left[\phi_{BA}^{B/A}\right]_{\times}\right) &= \exp\left(\theta\left[\mathbf{u}^{B/A}\right]_{\times}\right) \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\theta\left[\mathbf{u}^{B/A}\right]_{\times}\right)^k \\
&= \mathbf{I} + \theta\left[\mathbf{u}^{B/A}\right]_{\times} + \frac{\theta^2}{2}\left[\mathbf{u}^{B/A}\right]_{\times}^2 + \frac{\theta^3}{3!}\left[\mathbf{u}^{B/A}\right]_{\times}^3 + \dots \\
&= \mathbf{I} + \left(\theta - \frac{\theta^3}{3!} + \dots\right)\left[\mathbf{u}^{B/A}\right]_{\times} + \left(\frac{\theta^2}{2} - \frac{\theta^4}{4!} + \dots\right)\left[\mathbf{u}^{B/A}\right]_{\times}^2 \\
&= \mathbf{I} + \sin\theta\left[\mathbf{u}^{B/A}\right]_{\times} + (1 - \cos\theta)\left[\mathbf{u}^{B/A}\right]_{\times}^2 \\
&= \cos\theta\mathbf{I} + (1 - \cos\theta)\mathbf{u}^{B/A}\left(\mathbf{u}^{B/A}\right)^T + \sin\theta\left[\mathbf{u}^{B/A}\right]_{\times},
\end{aligned} \tag{130}$$

where we used the properties $\left[\mathbf{u}^{B/A}\right]_{\times}^2 = \mathbf{u}^{B/A}\left(\mathbf{u}^{B/A}\right)^T - \mathbf{I}$ and $\left[\mathbf{u}^{B/A}\right]_{\times}^3 = -\left[\mathbf{u}^{B/A}\right]_{\times}$. Equivalently, it can be rewritten as [HZ03]:

$$\begin{aligned}
\mathbf{R}_B^A &= \exp\left(\left[\phi_{BA}^{B/A}\right]_{\times}\right) \\
&= \mathbf{I} + \operatorname{sinc}\|\phi_{BA}^{B/A}\|\left[\phi_{BA}^{B/A}\right]_{\times} + \frac{1 - \cos\|\phi_{BA}^{B/A}\|}{\|\phi_{BA}^{B/A}\|^2}\left[\phi_{BA}^{B/A}\right]_{\times}^2 \\
&= \cos\|\phi_{BA}^{B/A}\|\mathbf{I} + \operatorname{sinc}\|\phi_{BA}^{B/A}\|\left[\phi_{BA}^{B/A}\right]_{\times} + \frac{1 - \cos\|\phi_{BA}^{B/A}\|}{\|\phi_{BA}^{B/A}\|^2}\phi_{BA}^{B/A}\left(\phi_{BA}^{B/A}\right)^T.
\end{aligned} \tag{131}$$

Equation (130) is the exact rotation formula from (127). Therefore, from (130), we can conclude:

$$\boxed{
\begin{aligned}
\mathbf{R}_B^A &= \exp\left(\left[\phi_{BA}^{B/A}\right]_{\times}\right), \\
\mathbf{R}_A^B &= \exp\left(-\left[\phi_{BA}^{B/A}\right]_{\times}\right),
\end{aligned}
} \tag{132}$$

where $\exp := \mathfrak{so}(3) \rightarrow SO(3)$ is called the *exponential map*. $\mathfrak{so}(3)$ is called a *Lie algebra* of $SO(3)$ and is discussed in Section 10.2.

As discussed before, the negative sign in (132) arises from the fact that \mathbf{R}_A^B is the inverse rotation of \mathbf{R}_B^A . The rotation vector $\vec{\phi}_{BA}$ encodes the orientation of frame F_B relative to frame F_A (encodes the rotation to apply to basis vectors of frame F_A to get the basis vectors of frame F_B). To inverse this rotation, a rotation described by the inverse rotation vector $\vec{\phi}_{AB} = -\vec{\phi}_{BA}$ needs to be applied. Since $\theta \in [0, \pi]$, the direction of the axis of rotation must be reversed, i.e., $-\vec{\mathbf{u}}$.

In general, it is common specify the resolving axis of a rotation vector to be identical to the frame orientation that is being described, i.e., ϕ_{BA}^B .

5.2 Rotation Matrix to Axis-Angle Representation

5.2.1 Logarithmic Map

As shown before, the rotation matrix from a rotation vector $\phi_{BA}^{B/A} \in \mathbb{R}^3$ can be obtained as:

$$\mathbf{R}_B^A = \mathbf{R} \left\{ \phi_{BA}^{B/A} \right\} = \begin{bmatrix} \cos \theta + u_1^2(1 - \cos \theta) & u_1 u_2(1 - \cos \theta) - u_3 \sin \theta & u_1 u_3(1 - \cos \theta) + u_2 \sin \theta \\ u_1 u_2(1 - \cos \theta) + u_3 \sin \theta & \cos \theta + u_2^2(1 - \cos \theta) & u_2 u_3(1 - \cos \theta) - u_1 \sin \theta \\ u_1 u_3(1 - \cos \theta) - u_2 \sin \theta & u_2 u_3(1 - \cos \theta) + u_1 \sin \theta & \cos \theta + u_3^2(1 - \cos \theta) \end{bmatrix}. \quad (133)$$

The angle of rotation, θ , can be computed from the trace of \mathbf{R}_B^A :

$$\begin{aligned} \text{tr}(\mathbf{R}_B^A) &= 3 \cos \theta + u_1^2(1 - \cos \theta) + u_2^2(1 - \cos \theta) + u_3^2(1 - \cos \theta) \\ &= 3 \cos \theta - (u_1^2 + u_2^2 + u_3^2) \cos \theta + (u_1^2 + u_2^2 + u_3^2) \\ &= 3 \cos \theta - \|\mathbf{u}^A\|^2 \cos \theta + \|\mathbf{u}^A\|^2 \\ &= 2 \cos \theta + 1. \end{aligned} \quad (134)$$

Hence, the angle of rotation is:

$$\theta = \arccos \left(\frac{\text{trace}(\mathbf{R}_B^A) - 1}{2} \right) \in \mathbb{R}. \quad (135)$$

The unit axis of rotation can be computed as:

$$\mathbf{u}^{B/A} = \frac{[\mathbf{R}_B^A - (\mathbf{R}_B^A)^T]^\vee}{2 \sin \theta} \in \mathbb{R}^3, \quad (136)$$

where $[\cdot]^\vee$ is the inverse of the skew operator $[\cdot]_\times$. These are called *logarithmic map* defined as:

$$\log : \mathbf{R} \in SO(3) \mapsto \log(\mathbf{R}) = [\theta \mathbf{u}]_\times \in \mathfrak{so}(3). \quad (137)$$

Logarithmic maps are discussed in Section 10.4.

5.3 Frame Invariance Property

As mentioned previously, the resolving frame does not matter for rotation vectors. We have:

$$\phi^B = \mathbf{R}_A^B \phi^A = \phi^A, \quad \mathbf{u}^B = \mathbf{R}_A^B \mathbf{u}^A = \mathbf{u}^A, \quad (138)$$

$$\phi^A = \mathbf{R}_B^A \phi^B = \phi^B, \quad \mathbf{u}^A = \mathbf{R}_B^A \mathbf{u}^B = \mathbf{u}^B. \quad (139)$$

This is called the *frame invariance property* of rotation vectors. To see why, let $\mathbf{u}^A, \mathbf{u}^B \in \mathbb{R}^3$ be the component vectors of $\bar{\mathbf{u}}$ in F_A and F_B , respectively. The component vectors are related by:

$$\mathbf{u}^B = \mathbf{R}_A^B \mathbf{u}^A = \mathbf{R} \left\{ -\phi_{BA}^A \right\} \mathbf{u}^A. \quad (140)$$

Substituting (128) into above yields:

$$\mathbf{u}^B = \mathbf{R}_A^B \mathbf{u}^A \quad (141)$$

$$= \left(\cos \theta \mathbf{I} + (1 - \cos \theta) \mathbf{u}^A (\mathbf{u}^A)^T - \sin \theta [\mathbf{u}^A]_\times \right) \mathbf{u}^A \quad (142)$$

$$= \cos \theta \mathbf{u}^A + (1 - \cos \theta) \mathbf{u}^A \quad (143)$$

$$= \mathbf{u}^A. \quad (144)$$

where we used the identities $\|\mathbf{u}^A\| = (\mathbf{u}^A)^T \mathbf{u}^A = 1$ and $[\mathbf{u}^A]_{\times} \mathbf{u}^A = \mathbf{u}^A \times \mathbf{u}^A = \mathbf{0}$. Hence, the scalar components of $\bar{\mathbf{u}}$ in frame F_A is identical to that of in frame F_B . We denote it as:

$$\boxed{\mathbf{u}^{B/A} \equiv \mathbf{u}^{B/A} := \mathbf{u}^A \equiv \mathbf{u}^B,} \quad (145)$$

and therefore, $\mathbf{u}^{B/A}$ is an eigenvector of \mathbf{R}_A^B and \mathbf{R}_B^A (associated with eigenvalue one). In fact, the eigen decomposition of the rotation matrix \mathbf{R}_A^B reveals one real eigenvalue of one, with its corresponding eigenvector representing the axis of rotation $\mathbf{u}^{B/A}$, and one complex conjugate pair, $\exp(\pm i\theta)$, which describe the rotation by angle θ in the plane perpendicular to the axis of rotation. These complex eigenvalues and their associated eigenvectors define the action of rotation, allowing decomposition into real-valued vectors that rotate within that plane, consistent with Euler's Rotation Theorem. Hence, $\bar{\mathbf{u}}$ is also called the *Euler axis*.

5.4 Infinitesimal Rotations

For small angle $\delta\theta \in \mathbb{R}^3$, we have:

$$\sin \delta\theta \approx \tan \delta\theta \approx \delta\theta, \quad \cos \delta\theta \approx 1. \quad (146)$$

Therefore, the rotation matrix described by the rotation vector $\delta\phi_{BA}^{B/A}$ can be approximated as:

$$\boxed{\begin{aligned} \mathbf{R}_B^A &\approx \mathbf{I} + [\delta\phi_{BA}^{B/A}]_{\times} = \mathbf{I} - [\delta\phi_{AB}^{A/B}]_{\times}, \\ \mathbf{R}_A^B &\approx \mathbf{I} + [\delta\phi_{AB}^{A/B}]_{\times} = \mathbf{I} - [\delta\phi_{BA}^{B/A}]_{\times}. \end{aligned}} \quad (147)$$

6 Quaternions

Let $\{q_0, q_1, q_2, q_3\} \in \mathbb{R}$ be a set of scalar coefficients and let $\{i, j, k\}$ be three imaginary unit numbers such that $i^2 = j^2 = k^2 = ijk = -1$. A *quaternion* is defined as:

$$\begin{aligned} \underline{q} &:= q_0 + iq_1 + jq_2 + kq_3 \\ &= q_0 + \underline{q}_v \in \mathbb{H}, \end{aligned} \quad (148)$$

where \mathbb{H} is the space of abstract quaternions. q_0 is called the *real* (or *scalar*) part and $\underline{q}_v := iq_1 + jq_2 + kq_3$ is called the *imaginary* part.

When a frame F_A is fixed, a quaternion $\underline{q} \in \mathbb{H}$ may be written as a mathematical column vector $\mathbf{q} \in \mathbb{R}^4$ in terms of its coefficients $\{q_0, q_1, q_2, q_3\}$, which allows us to use matrix algebra for operations involving quaternions. However, the ordering of the scalar and imaginary parts as well as the algebraic relationships between the imaginary unit numbers $\{i, j, k\}$ depends on the *quaternion convention*.

6.1 Hamilton vs. JPL Conventions

There are two different quaternion conventions: Hamilton and JPL quaternions, with the following differences [Sol17, TR05, SGB+18]:

1. *Handedness*: Hamilton convention uses *right-handed* quaternions ($k := ij$), whereas JPL convention uses *left-handed* quaternions ($k := -ij$). The left-handed quaternion can be converted to the right-handed quaternion by the *complex conjugate* operator:

$$\underline{q}_{\text{left}} = \underline{q}_{\text{right}}^* \quad (149)$$

2. *Components Ordering*: The mathematical column vector $\mathbf{q} \in \mathbb{R}^4$ has different ordering for the real and for the imaginary parts. Hamilton convention uses the ordering (q_0, \underline{q}_v) , whereas JPL convention uses the ordering (\underline{q}_v, q_0) .

The quaternion $q \in \mathbb{H}$ defined in (148) is an abstract quaternion and therefore convention-free. The Hamilton (denoted by ${}^H\mathbf{q}$) and the JPL (denoted by ${}^J\mathbf{q}$) quaternions are introduced only when we need their vector representations. Let $\mathbf{q}_v \in \mathbb{R}^3$ be the vector representation of the imaginary part \underline{q}_v . The vector representations of Hamilton and JPL quaternions therefore, can be written as:

$${}^H\mathbf{q} := \begin{bmatrix} q_0 \\ \mathbf{q}_v \end{bmatrix} \in \mathbb{R}^4, \quad {}^J\mathbf{q} := \begin{bmatrix} \mathbf{q}_v \\ q_0 \end{bmatrix} \in \mathbb{R}^4. \quad (150)$$

If the convention is clear from the context, we may simply write as \mathbf{q} .

Hamilton quaternions can be converted to JPL quaternions by re-ordering the components of the column vector \mathbf{q} followed by taking the complex conjugate via (149):

$${}^H\mathbf{q} = \begin{bmatrix} q_0 \\ \mathbf{q}_v \end{bmatrix} \xrightarrow{\text{re-order}} \begin{bmatrix} \mathbf{q}_v \\ q_0 \end{bmatrix} \xrightarrow{\text{conjugate}} {}^J\mathbf{q}. \quad (151)$$

Table 3 summarizes the key differences between the Hamilton and JPL conventions.

Quaternion Type	Hamilton	JPL
Components Order	${}^H\mathbf{q} := [q_0, \mathbf{q}_v]^T$	${}^J\mathbf{q} := [\mathbf{q}_v, q_0]^T$
Algebra	$k = ij = -ji,$ $i^2 = j^2 = k^2 = ijk = -1$	$k = ji = -ij,$ $i^2 = j^2 = k^2 = -1, ijk = 1$
Handedness	Right-handed	Left-handed

Table 3: Comparison of Hamilton and JPL Quaternion Conventions

6.2 Pure Quaternions and Vector Embedding

A *pure quaternion* is defined as:

$$\underline{q} := \underline{q}_v = iq_1 + jq_2 + kq_3 \in \text{Im}(\mathbb{H}) \subset \mathbb{H}. \quad (152)$$

Let F_A be a frame and $\vec{\mathbf{x}} \in \mathbb{V}$ a physical vector. Let $x_1, x_2, x_3 \in \mathbb{R}$ be the components of $\vec{\mathbf{x}}$ in F_A . The physical vector $\vec{\mathbf{x}}$ can be decomposed as:

$$\vec{\mathbf{x}} = x_1\hat{\mathbf{i}}_A + x_2\hat{\mathbf{j}}_A + x_3\hat{\mathbf{k}}_A. \quad (153)$$

Since \mathbb{V} and $\text{Im}(\mathbb{H})$ are finite-dimensional vector spaces with $\dim(\text{Im}(\mathbb{H})) = \dim(\mathbb{V}) = 3$, they are isomorphic as vector spaces: $\text{Im}(\mathbb{H}) \cong \mathbb{V}$. However, the isomorphism is not canonical; it depends on the choice of frame. Given a frame F_A , we define the linear bijection $\mathcal{Q}_A : \mathbb{V} \rightarrow \text{Im}(\mathbb{H})$ by:

$$\mathcal{Q}_A(\hat{\mathbf{i}}_A) = i, \quad \mathcal{Q}_A(\hat{\mathbf{j}}_A) = j, \quad \mathcal{Q}_A(\hat{\mathbf{k}}_A) = k. \quad (154)$$

By linearity, we have:

$$\begin{aligned}
\mathcal{Q}_A(\vec{\mathbf{x}}) &= \mathcal{Q}_A(x_1\hat{\mathbf{i}}_A + x_2\hat{\mathbf{j}}_A + x_3\hat{\mathbf{k}}_A) \\
&= x_1\mathcal{Q}_A(\hat{\mathbf{i}}_A) + x_2\mathcal{Q}_A(\hat{\mathbf{j}}_A) + x_3\mathcal{Q}_A(\hat{\mathbf{k}}_A) \\
&= x_1i + x_2j + x_3k \\
&= \begin{bmatrix} i & j & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
&= \mathcal{Q}_A(F_A)\mathbf{x}^A,
\end{aligned} \tag{155}$$

where we introduce the shorthand $\mathcal{Q}_A(F_A) := [i, j, k]$ to denote the quaternion basis associated with F_A vectrix.

The map $\mathcal{Q}_A : \mathbb{V} \rightarrow \text{Im}(\mathbb{H})$ sends a physical vector $\vec{\mathbf{x}} \in \mathbb{V}$ to the pure quaternion having the same components with respect to the quaternion basis $\{i, j, k\}$ as $\vec{\mathbf{x}}$ has in F_A . We define the pure quaternion representation of $\vec{\mathbf{x}}$ in F_A as:

$$\boxed{\vec{x} := \mathcal{Q}_A(\vec{\mathbf{x}}) = \mathcal{Q}_A(F_A)\mathbf{x}^A = x_1i + x_2j + x_3k.} \tag{156}$$

Its *vector embeddings* in frame F_A with Hamilton and JPL quaternions are defined as:

$$\boxed{{}^H\mathbf{x}^A = \begin{bmatrix} 0 \\ \mathbf{x}^A \end{bmatrix} \in \mathbb{R}^4, \quad {}^J\mathbf{x}^A = \begin{bmatrix} \mathbf{x}^A \\ 0 \end{bmatrix} \in \mathbb{R}^4.} \tag{157}$$

6.3 Quaternion Properties

6.3.1 Sum

The quaternion sum for the two conventions is shown in Table 4

Quaternion Property	Hamilton	JPL
Components order	${}^H\mathbf{q} = [q_0, \mathbf{q}_v]^T$	${}^J\mathbf{q} = [\mathbf{q}_v, q_0]^T$
Sum	$\mathbf{q} \pm \mathbf{p} = [q_0 \pm p_0, \mathbf{q}_v \pm \mathbf{p}_v]^T$	$\mathbf{q} \pm \mathbf{p} = [\mathbf{q}_v \pm \mathbf{p}_v, q_0 \pm p_0]^T$

Table 4: Quaternion Sum

By construction, the sum is *commutative* and *associative* for both conventions:

$$\mathbf{p} + \mathbf{q} = \mathbf{q} + \mathbf{p}, \tag{158}$$

$$\mathbf{p} + (\mathbf{q} + \mathbf{r}) = (\mathbf{p} + \mathbf{q}) + \mathbf{r}. \tag{159}$$

6.3.2 Quaternion Product

Due to the difference in handedness and component ordering, we use different quaternion product notations for Hamilton and JPL conventions. The quaternion products for Hamilton and JPL are denoted as:

$${}^H\mathbf{p} \otimes {}^H\mathbf{q}, \quad {}^H\mathbf{p}, {}^H\mathbf{q} \in \mathbb{R}^4, \quad {}^J\mathbf{p} \odot {}^J\mathbf{q}, \quad {}^J\mathbf{p}, {}^J\mathbf{q} \in \mathbb{R}^4. \tag{160}$$

Table 5 shows the main quaternion product properties of the two conventions.

Quaternion Property	Hamilton	JPL
Components order	${}^H\mathbf{q} = [q_0, \mathbf{q}_v]^T$	${}^J\mathbf{q} = [\mathbf{q}_v, q_0]^T$
Product	$\mathbf{q} \otimes \mathbf{p} = \begin{bmatrix} q_0 p_0 - \mathbf{q}_v^T \mathbf{p}_v \\ q_0 \mathbf{p}_v + p_0 \mathbf{q}_v + \mathbf{q}_v \times \mathbf{p}_v \end{bmatrix}$	$\mathbf{q} \odot \mathbf{p} = \begin{bmatrix} q_0 \mathbf{p}_v + p_0 \mathbf{q}_v - \mathbf{q}_v \times \mathbf{p}_v \\ q_0 p_0 - \mathbf{p}_v^T \mathbf{q}_v \end{bmatrix}$
Left Product Matrix	$[\mathbf{q}]_L = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix}$	$[\mathbf{q}]_L = \begin{bmatrix} q_0 & q_3 & -q_2 & q_1 \\ -q_3 & q_0 & q_1 & q_2 \\ q_2 & -q_1 & q_0 & q_3 \\ -q_1 & -q_2 & -q_3 & q_0 \end{bmatrix}$
Right Product Matrix	$[\mathbf{q}]_R = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & q_3 & -q_2 \\ q_2 & -q_3 & q_0 & q_1 \\ q_3 & q_2 & -q_1 & q_0 \end{bmatrix}$	$[\mathbf{q}]_R = \begin{bmatrix} q_0 & -q_3 & q_2 & q_1 \\ q_3 & q_0 & -q_1 & q_2 \\ -q_2 & q_1 & q_0 & q_3 \\ -q_1 & -q_2 & -q_3 & q_0 \end{bmatrix}$
Product using Product Matrix	$\mathbf{q} \otimes \mathbf{p} = [\mathbf{q}]_L \mathbf{p} = [\mathbf{p}]_R \mathbf{q}$	$\mathbf{q} \odot \mathbf{p} = [\mathbf{q}]_L \mathbf{p} = [\mathbf{p}]_R \mathbf{q}$

Table 5: Quaternion Product

Quaternion product is not commutative in the general case:

$$\mathbf{p} \otimes \mathbf{q} \neq \mathbf{q} \otimes \mathbf{p}, \quad \mathbf{p} \odot \mathbf{q} \neq \mathbf{q} \odot \mathbf{p}. \quad (161)$$

Exceptions to this general non-commutativity are limited to the cases where $\mathbf{p}_v \times \mathbf{q}_v = 0$, which happens whenever one quaternion is real such that $\mathbf{p} = p_0$ or $\mathbf{q} = q_0$, or when both vector parts are parallel, $\mathbf{p}_v \parallel \mathbf{q}_v$.

The quaternion product is associative and distributive over the sum:

$$(\mathbf{p} \otimes \mathbf{q}) \otimes \mathbf{r} = \mathbf{p} \otimes (\mathbf{q} \otimes \mathbf{r}), \quad (\mathbf{p} \odot \mathbf{q}) \odot \mathbf{r} = \mathbf{p} \odot (\mathbf{q} \odot \mathbf{r}), \quad (162)$$

$$\mathbf{p} \otimes (\mathbf{q} + \mathbf{r}) = \mathbf{p} \otimes \mathbf{q} + \mathbf{p} \otimes \mathbf{r}, \quad \mathbf{p} \odot (\mathbf{q} + \mathbf{r}) = \mathbf{p} \odot \mathbf{q} + \mathbf{p} \odot \mathbf{r}. \quad (163)$$

6.3.3 Identity, Conjugate, and Inverse

Table 6 summarizes the identity quaternion, conjugate of a quaternion, and inverse of a quaternion for the two conventions.

Quaternion Property	Hamilton	JPL
Components order	${}^H\mathbf{q} = [q_0, \mathbf{q}_v]^T$	${}^J\mathbf{q} = [\mathbf{q}_v, q_0]^T$
Identity	$\mathbf{q}_0 = [1, \mathbf{0}_v]^T$, $\mathbf{q}_0 \otimes \mathbf{q} = \mathbf{q} \otimes \mathbf{q}_0 = \mathbf{q}$	$\mathbf{q}_0 = [\mathbf{0}_v, 1]^T$, $\mathbf{q}_0 \odot \mathbf{q} = \mathbf{q} \odot \mathbf{q}_0 = \mathbf{q}$
Conjugate	$\mathbf{q}^* := [q_0, -\mathbf{q}_v]^T$	$\mathbf{q}^* := [-\mathbf{q}_v, q_0]^T$
Inverse	$\mathbf{q}^{-1} := \mathbf{q}^* / \ \mathbf{q}\ ^2$	$\mathbf{q}^{-1} := \mathbf{q}^* / \ \mathbf{q}\ ^2$

Table 6: Identity, Conjugate, and Inverse

The conjugate and the inverse operations have the following distributive properties:

$$(\mathbf{q} \otimes \mathbf{p})^* = \mathbf{p}^* \otimes \mathbf{q}^*, \quad (\mathbf{q} \odot \mathbf{p})^* = \mathbf{p}^* \odot \mathbf{q}^*, \quad (164)$$

$$(\mathbf{q} \otimes \mathbf{p})^{-1} = \mathbf{p}^{-1} \otimes \mathbf{q}^{-1}, \quad (\mathbf{q} \odot \mathbf{p})^{-1} = \mathbf{p}^{-1} \odot \mathbf{q}^{-1}. \quad (165)$$

6.3.4 Norm

The norm of a quaternion is defined as:

$$\|\underline{q}\| = \sqrt{q_0^2 + \|\mathbf{q}_v\|^2} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} \in \mathbb{R}. \quad (166)$$

Similarly, for the vector representations, we have:

$$\|{}^H\mathbf{q}\| := \sqrt{{}^H\mathbf{q} \otimes {}^H\mathbf{q}^*} = \sqrt{{}^H\mathbf{q}^* \otimes {}^H\mathbf{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} \in \mathbb{R}, \quad (167)$$

$$\|{}^J\mathbf{q}\| := \sqrt{{}^J\mathbf{q} \odot {}^J\mathbf{q}^*} = \sqrt{{}^J\mathbf{q}^* \odot {}^J\mathbf{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} \in \mathbb{R}. \quad (168)$$

6.3.5 Unit Quaternions

Unit quaternions are a set of quaternions with unit norm. Unit quaternions have the property:

$$\mathbf{q}^{-1} = \mathbf{q}^*. \quad (169)$$

Let F_A be a frame. Let $\rho \in \mathbb{R}$ be a scalar angle and $\mathbf{u} = [u_1, u_2, u_3] \in \mathbb{R}^3$ be a unit component vector such that $\|\mathbf{u}\| = 1$. A unit quaternion can be expressed by a vector $\rho\mathbf{u}$ such that:

$${}^H\mathbf{q} = \begin{bmatrix} \cos \rho \\ \sin \rho \mathbf{u} \end{bmatrix}, \quad {}^J\mathbf{q} = \begin{bmatrix} \sin \rho \mathbf{u} \\ \cos \rho \end{bmatrix}. \quad (170)$$

To see this, let $\underline{q} = q_0 + \underline{q}_v$ be a unit quaternion. Since \underline{q} has a unit norm, we have:

$$\|\underline{q}\|^2 = q_0^2 + \|\mathbf{q}_v\|^2 = 1. \quad (171)$$

Since $q_0 \in [-1, 1]$, there exists $\rho \in [0, \pi]$ such that $q_0 = \cos \rho$. Substituting into (171) gives:

$$\|\mathbf{q}_v\|^2 = 1 - \cos^2 \rho = \sin^2 \rho. \quad (172)$$

Let $\vec{u} \in \text{Im}(\mathbb{H})$ be a unit pure quaternion such that $\|\vec{u}\| = 1$. Define the coefficients $\{u_1, u_2, u_3\}$ of \vec{u} as:

$$u_1 = \frac{q_1}{\|\mathbf{q}_v\|}, \quad u_2 = \frac{q_2}{\|\mathbf{q}_v\|}, \quad u_3 = \frac{q_3}{\|\mathbf{q}_v\|}, \quad (173)$$

assuming $\mathbf{q}_v \neq \mathbf{0}$. Hence, we have:

$$\underline{q} = q_0 + \underline{q}_v \quad (174)$$

$$= \cos \rho + \sin \rho \vec{u}, \quad (175)$$

and its vector representations are:

$$\mathbb{H}_{\mathbf{q}} = \begin{bmatrix} \cos \rho \\ \sin \rho \mathbf{u} \end{bmatrix}, \quad \mathbb{J}_{\mathbf{q}} = \begin{bmatrix} \sin \rho \mathbf{u} \\ \cos \rho \end{bmatrix}, \quad (176)$$

where $\mathbf{u} \in \mathbb{R}^3$ is the vector representation of \vec{u} .

6.4 Quaternion Exponentials and Logarithms

The quaternion exponential is a function on quaternions analogous to the ordinary exponential function. It is defined as:

$$\exp(\underline{q}) := \sum_{k=0}^{\infty} \frac{1}{k!} \underline{q}^k \in \mathbb{H}. \quad (177)$$

6.4.1 Pure Quaternion Exponentials

Let $\underline{q} = \underline{q}_v = q_1 i + q_2 j + q_3 k \in \text{Im}(\mathbb{H})$ be a pure quaternion. Let $\rho = \|\underline{q}_v\| \in \mathbb{R}$ be a scalar angle and let $\vec{u} = u_1 i + u_2 j + u_3 k \in \text{Im}(\mathbb{H})$ with $\|\vec{u}\| = 1$ such that $\underline{q} = \rho \vec{u}$. The exponential of a pure quaternion can be computed as:

$$\begin{aligned} \exp(\underline{q}) &= \exp(\rho \vec{u}) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} (\underline{q}_v)^k \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} (\rho \vec{u})^k \\ &= \left(1 - \frac{\rho^2}{2!} + \frac{\rho^4}{4!} + \dots\right) + \left(\rho \vec{u} - \frac{\rho^3 \vec{u}}{3!} + \frac{\rho^5 \vec{u}}{5!} + \dots\right) \\ &= \cos \rho + \sin \rho \vec{u}, \end{aligned} \quad (178)$$

which is the extension of the Euler's formula, $\exp(i\theta) = \cos \theta + i \sin \theta$, defined for complex numbers.

6.4.2 Quaternion Exponentials

For a generic quaternion $\underline{q} \in \mathbb{H}$, the quaternion exponential can be computed as:

$$\begin{aligned} \exp(\underline{q}) &= \exp(q_0 + \underline{q}_v) \\ &= \exp(q_0) \exp(\underline{q}_v), \end{aligned} \quad (179)$$

since commutativity holds. The pure quaternion exponential can be computed by (178). Hence, we have:

$$\exp(\underline{q}) = \exp(q_0) \exp\left(\frac{\underline{q}_v}{\|\underline{q}_v\|}\right) \quad (180)$$

$$= \exp(q_0) (\cos \rho + \vec{u} \sin \rho) \quad (181)$$

$$= \exp(q_0) \left(\cos \|\underline{q}_v\| + \frac{\underline{q}_v}{\|\underline{q}_v\|} \sin \|\underline{q}_v\| \right). \quad (182)$$

6.4.3 Pure Quaternion Logarithms

6.4.4 Quaternion Logarithms

6.5 Unit Quaternion Realization of $SO(3)$

As discussed in Section 3.2.2, the set of unit quaternions forms a group under the operation of quaternion composition. This group is topologically a 3-sphere denoted as $S^3 \subset \mathbb{H}$:

$$S^3 = \{\underline{q} \in \mathbb{H} \mid \|\underline{q}\| = 1\}. \quad (183)$$

Let F_A be a frame and let $\mathbf{x}^A \in \mathbb{R}^3$ be a component vector. Let $\mathbf{R}_1, \mathbf{R}_2 \in SO(3)$ be rotation matrices. The rotation operations by \mathbf{R}_1 and \mathbf{R}_2 can be represented by the Hamilton and JPL quaternion rotation actions defined as:

$$\begin{aligned} \mathbf{y}^A = \mathbf{R}_1 \mathbf{x}^A &\iff \mathbf{H} \mathbf{y}^A = \mathbf{H} \mathbf{q} \otimes \mathbf{H} \mathbf{x}^A \otimes \mathbf{H} \mathbf{q}^* \\ \mathbf{z}^A = \mathbf{R}_2 \mathbf{x}^A &\iff \mathbf{J} \mathbf{z}^A = \mathbf{J} \mathbf{q} \odot \mathbf{J} \mathbf{x}^A \odot \mathbf{J} \mathbf{q}^*, \end{aligned} \quad (184)$$

where $\mathbf{H} \mathbf{y}^A, \mathbf{J} \mathbf{z}^A \in \mathbb{R}^4$ are the vector embeddings of the component vectors $\mathbf{y}^A, \mathbf{z}^A \in \mathbb{R}^3$ using Hamilton and JPL quaternions, respectively. The reason why the rotation matrices and the rotated vectors are different are discussed shortly.

To see that the quaternion rotation actions satisfy the properties of $SO(3)$, we verify the orthogonality condition as:

$$\begin{aligned} \|\mathbf{H} \mathbf{q} \otimes \mathbf{H} \mathbf{x}^A \otimes \mathbf{H} \mathbf{q}^*\| &= \|\mathbf{H} \mathbf{q}\|^2 \|\mathbf{H} \mathbf{x}^A\| = \|\mathbf{H} \mathbf{x}^A\|, \\ \|\mathbf{J} \mathbf{q} \odot \mathbf{J} \mathbf{x}^A \odot \mathbf{J} \mathbf{q}^*\| &= \|\mathbf{J} \mathbf{q}\|^2 \|\mathbf{J} \mathbf{x}^A\| = \|\mathbf{J} \mathbf{x}^A\|, \end{aligned} \quad (185)$$

which implies that $\|\mathbf{H} \mathbf{q}\|^2 = \|\mathbf{J} \mathbf{q}\|^2 = 1$. Therefore:

$$\mathbf{H} \mathbf{q}^* \otimes \mathbf{H} \mathbf{q} = \mathbf{H} \mathbf{q} \otimes \mathbf{H} \mathbf{q}^* = 1, \quad \mathbf{J} \mathbf{q}^* \odot \mathbf{J} \mathbf{q} = \mathbf{J} \mathbf{q} \odot \mathbf{J} \mathbf{q}^* = 1, \quad (186)$$

which is akin to the orthogonality condition for rotation matrices, i.e., $\mathbf{R}^T \mathbf{R} = \mathbf{R} \mathbf{R}^T = \mathbf{I}$. The relative orientation conditions can be easily verified similarly.

Since S^3 is a double cover of $SO(3)$, we have:

$$\boxed{SO(3) \cong S^3 / \pm 1}, \quad (187)$$

meaning $SO(3)$ is obtained from S^3 by identifying the antipodal points — exactly the same identification as we saw with the ball model.

6.5.1 Quaternion and Rotation Vector Relationship

Let F_A and F_B be frames and $\vec{\phi}_{BA} \in \mathbb{V}$ be a rotation vector of F_B relative to F_A . Let $\phi_{BA}^{B/A} = \theta \mathbf{u}^{B/A} \in \mathbb{R}^3$ be the component vector of the rotation vector $\vec{\phi}_{BA}$. Since S^3 is the double cover of $SO(3)$, rotation by θ in $SO(3)$ is equivalent to a rotation by half the angle in S^3 such that:

$$\rho := \theta/2, \quad (188)$$

as illustrated in Figure X. Hence, the corresponding unit quaternion representation is:

$$\underline{q} = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \vec{u}, \quad (189)$$

where \vec{u} is the unit pure quaternion with $\mathbf{u}^{B/A}$ as coefficients. From (170), the equivalent vector representations are given as:

$${}^H\mathbf{q}\{\phi_{BA}^{B/A}\} = \begin{bmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \mathbf{u}^{B/A} \end{bmatrix}, \quad {}^J\mathbf{q}\{\phi_{BA}^{B/A}\} = \begin{bmatrix} \sin \frac{\theta}{2} \mathbf{u}^{B/A} \\ \cos \frac{\theta}{2} \end{bmatrix}. \quad (190)$$

To understand the rotation operation with Hamilton and JPL quaternions, we can look at vector rotations using the two conventions.

6.5.2 Active Rotation with Hamilton Quaternions

Let F_A be a frame and $\vec{x} \in \mathbb{V}$ a physical vector. Let $\phi = \theta \mathbf{u} \in \mathbb{R}^3$ be a component vector of a rotation vector and define $\vec{x} := \mathcal{Q}_A(F_A) \mathbf{x}^A = [i, j, k] \mathbf{x}^A$. Consider a rotation of \vec{x} by a rotation operation represented by a rotation vector $\vec{\phi}$ or equivalently by a unit quaternion $\underline{q} = \cos(\theta/2) + \sin(\theta/2) \vec{u}$.

The rotation operation by \underline{q} and its Hamiltonian vector representation $\mathbf{q} := {}^H\mathbf{q} \in \mathbb{R}^4$ is defined as:

$$\mathbf{R}\{\mathbf{q}\} \mathbf{x}^A := \mathbf{q} \otimes {}^H\mathbf{x}^A \otimes \mathbf{q}^*. \quad (191)$$

Let the rotated component vector be $\mathbf{y}^A \in \mathbb{R}^3$. We have:

$$\begin{aligned} {}^H\mathbf{y}^A &= \mathbf{q} \otimes {}^H\mathbf{x}^A \otimes \mathbf{q}^* \\ &= \begin{bmatrix} \cos \frac{\theta}{2} \\ \mathbf{u} \sin \frac{\theta}{2} \end{bmatrix} \otimes \begin{bmatrix} 0 \\ \mathbf{x}^A \end{bmatrix} \otimes \begin{bmatrix} \cos \frac{\theta}{2} \\ -\mathbf{u} \sin \frac{\theta}{2} \end{bmatrix} \\ &= \begin{bmatrix} -\sin \frac{\theta}{2} \mathbf{u}^T \mathbf{x}^A \\ \cos \frac{\theta}{2} \mathbf{x}^A + \sin \frac{\theta}{2} \mathbf{u} \times \mathbf{x}^A \end{bmatrix} \otimes \begin{bmatrix} \cos \frac{\theta}{2} \\ -\mathbf{u} \sin \frac{\theta}{2} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ \cos^2 \frac{\theta}{2} \mathbf{x}^A + 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} (\mathbf{u} \times \mathbf{x}^A) + 2 \sin^2 \frac{\theta}{2} (\mathbf{u}^T \mathbf{x}^A) \mathbf{u} - \sin^2 \frac{\theta}{2} \mathbf{x}^A \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ \mathbf{x}^A \cos \theta + (\mathbf{u} \times \mathbf{x}^A) \sin \theta + \mathbf{u} (\mathbf{u}^T \mathbf{x}^A) (1 - \cos \theta) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ \mathbf{u} (\mathbf{u}^T \mathbf{x}^A) + (\mathbf{x}^A - \mathbf{u} \mathbf{u}^T \mathbf{x}^A) \cos \theta + (\mathbf{u} \times \mathbf{x}^A) \sin \theta \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ (\cos \theta \mathbf{I} + (1 - \cos \theta) \mathbf{u} \mathbf{u}^T + \sin \theta [\mathbf{u}]_{\times}) \mathbf{x}^A \end{bmatrix}, \end{aligned} \quad (192)$$

which is the *Rodrigues'* vector rotation formula obtained in (46).

6.5.3 Active Rotation with JPL Quaternions

Let the rotated component vector be $\mathbf{z}^A \in \mathbb{R}^3$. The rotation operation by $\mathbf{q} := {}^J\mathbf{q} \in \mathbb{R}^4$ is defined as:

$$\begin{aligned}
{}^J\mathbf{z}^A &= \mathbf{q} \odot {}^J\mathbf{x}^A \odot \mathbf{q}^* \\
&= \begin{bmatrix} \mathbf{u} \sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{bmatrix} \odot \begin{bmatrix} \mathbf{x}^A \\ 0 \end{bmatrix} \odot \begin{bmatrix} -\mathbf{u} \sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{bmatrix} \\
&= \begin{bmatrix} \cos \frac{\theta}{2} \mathbf{x}^A - \sin \frac{\theta}{2} (\mathbf{u} \times \mathbf{x}^A) \\ -\sin \frac{\theta}{2} \mathbf{u}^T \mathbf{x}^A \end{bmatrix} \odot \begin{bmatrix} -\mathbf{u} \sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{bmatrix} \\
&= \begin{bmatrix} \cos^2 \frac{\theta}{2} \mathbf{x}^A - 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} (\mathbf{u} \times \mathbf{x}^A) + 2 \sin^2 \frac{\theta}{2} (\mathbf{u}^T \mathbf{x}^A) \mathbf{u} - \sin^2 \frac{\theta}{2} \mathbf{x}^A \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{x}^A \cos \theta - (\mathbf{u} \times \mathbf{x}^A) \sin \theta + \mathbf{u} (\mathbf{u}^T \mathbf{x}^A) (1 - \cos \theta) \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ \mathbf{u} (\mathbf{u}^T \mathbf{x}^A) + (\mathbf{x}^A - \mathbf{u} \mathbf{u}^T \mathbf{x}^A) \cos \theta - (\mathbf{u} \times \mathbf{x}^A) \sin \theta \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ (\cos \theta \mathbf{I} + (1 - \cos \theta) \mathbf{u} \mathbf{u}^T - \sin \theta [\mathbf{u}]_{\times}) \mathbf{x}^A \end{bmatrix}, \tag{193}
\end{aligned}$$

which is the inverse rotation defined by the *Rodrigues'* vector rotation formula obtained in (46).

6.6 Quaternion to Rotation Matrix

Let F_A and F_B be frames with $\{\hat{\mathbf{i}}_A, \hat{\mathbf{j}}_A, \hat{\mathbf{k}}_A\}$, $\{\hat{\mathbf{i}}_B, \hat{\mathbf{j}}_B, \hat{\mathbf{k}}_B\}$ basis vectors, respectively. Let $\mathbf{R}_B^A \in SO(3)$ be a rotation matrix and let $\vec{\phi}_{BA} \in \mathbb{V}$ be a rotation vector of F_B relative to F_A . As discussed previously, given the component vector $\phi_{BA}^{B/A} = \theta \mathbf{u}^{B/A} \in \mathbb{R}^3$, the rotation vector and rotation matrix have the following relationship:

$$\boxed{\mathbf{R}_B^A = \mathbf{R} \left\{ \phi_{BA}^{B/A} \right\} = \mathbf{R} \left\{ \theta \mathbf{u}^{B/A} \right\}.} \tag{194}$$

The equivalent rotation operation is parameterized by the unit quaternion defined as:

$$\underline{q}_B^A := \underline{q} \{ \vec{\phi}_{BA} \} \in S^3. \tag{195}$$

However, its vector representation and the corresponding transformation to a rotation matrix, $\mathbf{R} := \mathbf{R}\{\mathbf{q}\}$, is convention dependent.

6.6.1 Hamilton Quaternion to Rotation Matrix

Using the Hamilton quaternions $\mathbf{q} := {}^H\mathbf{q}$, the basis vectors of F_B can be obtained by rotating the basis vectors of F_A :

$$\begin{aligned}
{}^H\hat{\mathbf{i}}_A^B &= \mathbf{q} \otimes {}^H\hat{\mathbf{i}}_A^A \otimes \mathbf{q}^* = [\mathbf{q}]_L [\mathbf{q}^*]_R \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ q_0^2 + q_1^2 - q_2^2 - q_3^2 \\ 2(q_1q_2 + q_0q_3) \\ 2(q_1q_3 - q_0q_2) \end{bmatrix}, \\
{}^H\hat{\mathbf{j}}_A^B &= \mathbf{q} \otimes {}^H\hat{\mathbf{j}}_A^A \otimes \mathbf{q}^* = [\mathbf{q}]_L [\mathbf{q}^*]_R \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2(q_1q_2 - q_0q_3) \\ q_0^2 - q_1^2 + q_2^2 - q_3^2 \\ 2(q_2q_3 + q_0q_1) \end{bmatrix}, \\
{}^H\hat{\mathbf{k}}_A^B &= \mathbf{q} \otimes {}^H\hat{\mathbf{k}}_A^A \otimes \mathbf{q}^* = [\mathbf{q}]_L [\mathbf{q}^*]_R \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2(q_1q_3 + q_0q_2) \\ 2(q_2q_3 - q_0q_1) \\ q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}.
\end{aligned} \tag{196}$$

Rewriting in matrix form with the vectrices yields the rotation matrix:

$$\mathcal{F}_B = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 + q_0q_3) & 2(q_1q_3 - q_0q_2) \\ 2(q_1q_2 - q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 + q_0q_1) \\ 2(q_1q_3 + q_0q_2) & 2(q_2q_3 - q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} \mathcal{F}_A, \tag{197}$$

or equivalently:

$$\mathcal{F}_B = \mathcal{F}_A \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}. \tag{198}$$

Hence, from (90), we have:

$$\boxed{
\begin{aligned}
\mathbf{R} \{ {}^H\mathbf{q} \} &= \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}. \\
&= (q_0^2 - \mathbf{q}_v^T \mathbf{q}_v) \mathbf{I} + 2\mathbf{q}_v \mathbf{q}_v^T + 2q_0 [\mathbf{q}_v]_\times
\end{aligned}
} \tag{199}$$

6.6.2 JPL Quaternion to Rotation Matrix

Using the JPL quaternions $\mathbf{q} := {}^J\mathbf{q}$, the basis vectors of F_B can be obtained by rotating the basis vectors of F_A . However, it gives the inverse rotation matrix of the Hamilton rotation matrix obtained by (199).

To see this, we rotate the basis vectors of F_A as follows:

$$\begin{aligned}
{}^J\hat{\mathbf{i}}_A^B &= \mathbf{q} \odot {}^J\hat{\mathbf{i}}_A^A \odot \mathbf{q}^* = [\mathbf{q}]_L [\mathbf{q}^*]_R \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 \\ 2(q_1q_2 - q_0q_3) \\ 2(q_1q_3 + q_0q_2) \\ 0 \end{bmatrix}, \\
{}^J\hat{\mathbf{j}}_A^B &= \mathbf{q} \odot {}^J\hat{\mathbf{j}}_A^A \odot \mathbf{q}^* = [\mathbf{q}]_L [\mathbf{q}^*]_R \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2(q_1q_2 + q_0q_3) \\ q_0^2 - q_1^2 + q_2^2 - q_3^2 \\ 2(q_2q_3 - q_0q_1) \\ 0 \end{bmatrix}, \\
{}^J\hat{\mathbf{k}}_A^B &= \mathbf{q} \odot {}^J\hat{\mathbf{k}}_A^A \odot \mathbf{q}^* = [\mathbf{q}]_L [\mathbf{q}^*]_R \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2(q_1q_3 - q_0q_2) \\ 2(q_2q_3 + q_0q_1) \\ q_0^2 - q_1^2 - q_2^2 + q_3^2 \\ 0 \end{bmatrix}.
\end{aligned} \tag{200}$$

Rewriting in matrix form with the vectrices yields the rotation matrix:

$$\mathcal{F}_B = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} \mathcal{F}_A, \tag{201}$$

or equivalently:

$$F_B = F_A \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 + q_0q_3) & 2(q_1q_3 - q_0q_2) \\ 2(q_1q_2 - q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 + q_0q_1) \\ 2(q_1q_3 + q_0q_2) & 2(q_2q_3 - q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}. \tag{202}$$

Hence, from (90), we have:

$$\mathbf{R} \{ {}^J\mathbf{q} \} = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 + q_0q_3) & 2(q_1q_3 - q_0q_2) \\ 2(q_1q_2 - q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 + q_0q_1) \\ 2(q_1q_3 + q_0q_2) & 2(q_2q_3 - q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} \tag{203}$$

$$= (q_0^2 - \mathbf{q}_v^T \mathbf{q}_v) + 2\mathbf{q}_v \mathbf{q}_v^T - 2q_0 [\mathbf{q}_v]_{\times},$$

which is the inverse of the rotation matrix obtained in (199). The relationship between Hamilton and JPL quaternions is discussed in Section 6.8.

6.7 Rotation Matrix to Quaternion

Given a rotation matrix $\mathbf{R}_B^A \in SO(3)$, we seek the corresponding unit quaternion $q_B^A \in S^3$. Since S^3 is the double cover of $SO(3)$, there exist exactly two quaternions $\pm q_B^A$ that map to the same rotation matrix. By convention, we choose the quaternion with non-negative scalar part $q_0 \geq 0$.

6.7.1 Hamilton Quaternion from Rotation Matrix

Let $\mathbf{q} := {}^H\mathbf{q}_B^A = [q_0, q_1, q_2, q_3]^T \in \mathbb{R}^4$ be the Hamilton quaternion and let R_{ij} denote the (i, j) entry of $\mathbf{R}_B^A = \mathbf{R}\{{}^H\mathbf{q}_B^A\}$. From the diagonal entries of \mathbf{R}_B^A in (199), using the unit norm constraint $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$, we have:

$$\begin{aligned} 1 + \text{tr}(\mathbf{R}_B^A) &= 4q_0^2, \\ 1 + R_{11} - R_{22} - R_{33} &= 4q_1^2, \\ 1 - R_{11} + R_{22} - R_{33} &= 4q_2^2, \\ 1 - R_{11} - R_{22} + R_{33} &= 4q_3^2. \end{aligned} \tag{204}$$

From the antisymmetric identities of the off-diagonal elements, we get:

$$\begin{aligned} R_{23} - R_{32} &= 4q_0q_1, \\ R_{31} - R_{13} &= 4q_0q_2, \\ R_{12} - R_{21} &= 4q_0q_3. \end{aligned} \tag{205}$$

From the symmetric identities of the off-diagonal elements, we get:

$$\begin{aligned} R_{12} + R_{21} &= 4q_1q_2, \\ R_{13} + R_{31} &= 4q_1q_3, \\ R_{23} + R_{32} &= 4q_2q_3. \end{aligned} \tag{206}$$

When $q_0 \neq 0$ (i.e., $\theta \neq \pi$), the quaternion coefficients can be directly extracted using (204) and (205):

$$\boxed{\begin{aligned} q_0 &= \frac{1}{2}\sqrt{1 + \text{tr}(\mathbf{R}_B^A)}, \\ q_1 &= \frac{R_{23} - R_{32}}{4q_0}, \quad q_2 = \frac{R_{31} - R_{13}}{4q_0}, \quad q_3 = \frac{R_{12} - R_{21}}{4q_0}. \end{aligned}} \tag{207}$$

However, (207) is numerically unstable when $q_0 \approx 0$ (i.e., $\theta \approx \pi$) due to division by a near-zero value.

Shepperd's Method *Shepperd's method* [She78] avoids numerical instability by always dividing by the largest quaternion component. From (204), define:

$$\begin{aligned} t_0 &:= 1 + \text{tr}(\mathbf{R}_B^A) = 4q_0^2, \\ t_1 &:= 1 + R_{11} - R_{22} - R_{33} = 4q_1^2, \\ t_2 &:= 1 - R_{11} + R_{22} - R_{33} = 4q_2^2, \\ t_3 &:= 1 - R_{11} - R_{22} + R_{33} = 4q_3^2. \end{aligned} \tag{208}$$

Find $k = \arg \max_i t_i$. The largest t_k corresponds to the largest $|q_k|$, ensuring a numerically stable division. Then compute $q_k = \frac{1}{2}\sqrt{t_k}$ and obtain the remaining components by dividing by $4q_k$ using (205)–(206):

$$\boxed{\begin{array}{llll} \text{Case } t_0 \text{ largest:} & q_0 = \frac{1}{2}\sqrt{t_0}, & q_1 = \frac{R_{23} - R_{32}}{4q_0}, & q_2 = \frac{R_{31} - R_{13}}{4q_0}, & q_3 = \frac{R_{12} - R_{21}}{4q_0}, \\ \text{Case } t_1 \text{ largest:} & q_1 = \frac{1}{2}\sqrt{t_1}, & q_0 = \frac{R_{23} - R_{32}}{4q_1}, & q_2 = \frac{R_{12} + R_{21}}{4q_1}, & q_3 = \frac{R_{13} + R_{31}}{4q_1}, \\ \text{Case } t_2 \text{ largest:} & q_2 = \frac{1}{2}\sqrt{t_2}, & q_0 = \frac{R_{31} - R_{13}}{4q_2}, & q_1 = \frac{R_{12} + R_{21}}{4q_2}, & q_3 = \frac{R_{23} + R_{32}}{4q_2}, \\ \text{Case } t_3 \text{ largest:} & q_3 = \frac{1}{2}\sqrt{t_3}, & q_0 = \frac{R_{12} - R_{21}}{4q_3}, & q_1 = \frac{R_{13} + R_{31}}{4q_3}, & q_2 = \frac{R_{23} + R_{32}}{4q_3}. \end{array}} \tag{209}$$

If the convention $q_0 \geq 0$ is desired, negate all four components when $q_0 < 0$.

6.7.2 JPL Quaternion from Rotation Matrix

For the JPL quaternion $\mathbf{q} := {}^J\mathbf{q}_B^A = [q_0, q_1]^T$, the JPL rotation matrix $\mathbf{R}_B^A = \mathbf{R}\{{}^J\mathbf{q}_B^A\}$ is the transpose of the Hamilton rotation matrix for the same quaternion components q_0, q_1, q_2, q_3 . Since the diagonal entries are unchanged, the identities in (204) and the symmetric identities in (206) remain the same. However, the antisymmetric identities (205) acquire a sign change:

$$\begin{aligned} R_{32} - R_{23} &= 4q_0q_1, \\ R_{13} - R_{31} &= 4q_0q_2, \\ R_{21} - R_{12} &= 4q_0q_3. \end{aligned} \tag{210}$$

Hence, Shepperd's method for JPL quaternions follows the same structure as (209), with the antisymmetric terms $(R_{ij} - R_{ji})$ replaced by $(R_{ji} - R_{ij})$.

6.8 Interpretations of Hamilton and JPL Quaternions

The vector rotations using Hamilton (192) and JPL (193) quaternions produced different rotated vectors:

$$\begin{aligned} {}^H\mathbf{y}^A &= \mathbf{q} \otimes {}^H\mathbf{x}^A \otimes \mathbf{q}^* \\ &= \left[\begin{array}{c} 0 \\ \underbrace{(\cos \theta \mathbf{I} + (1 - \cos \theta) \mathbf{u}\mathbf{u}^T + \sin \theta [\mathbf{u}]_{\times})}_{\mathbf{R}_B^A} \mathbf{x}^A \end{array} \right], \\ {}^J\mathbf{z}^A &= \mathbf{q} \odot {}^J\mathbf{x}^A \odot \mathbf{q}^* \\ &= \left[\begin{array}{c} 0 \\ \underbrace{(\cos \theta \mathbf{I} + (1 - \cos \theta) \mathbf{u}\mathbf{u}^T - \sin \theta [\mathbf{u}]_{\times})}_{\mathbf{R}_A^B} \mathbf{x}^A \end{array} \right]. \end{aligned} \tag{211}$$

Comparing the rotation operation actions with results from Section 5.1 shows that given the same rotation vector $\phi_{BA}^{B/A}$, the Hamilton and JPL produces inverse rotations: the frame F_B obtained in (203) using JPL quaternions is not the same F_B obtained in (199) using Hamilton conventions.

Therefore, given frames F_A and F_B with the same underlying rotation parameterized by $\phi_{BA}^{B/A}$, we have the following relationship:

$$\begin{aligned} \mathbf{R}_B^A &:= \mathbf{R}\left\{{}^H\mathbf{q}\{\phi_{BA}^{B/A}\}\right\} = \mathbf{R}\left\{{}^J\mathbf{q}\{-\phi_{BA}^{B/A}\}\right\}, \\ \mathbf{R}_A^B &:= \mathbf{R}\left\{{}^J\mathbf{q}\{\phi_{BA}^{B/A}\}\right\} = \mathbf{R}\left\{{}^H\mathbf{q}\{-\phi_{BA}^{B/A}\}\right\}. \end{aligned} \tag{212}$$

This is often the most common source of confusion in quaternion based attitude estimation. We define $\underline{q}_B^A \in \mathbb{H}$ with $\|\underline{q}\| = 1$ as the orientation of frame F_B relative to frame F_A in unit quaternion representation. Its vector representation is $\mathbf{q}_B^A \in \mathbb{R}^4$ and is convention dependent:

$$\boxed{{}^H\mathbf{q}_B^A := {}^H\mathbf{q}\{\phi_{BA}^{B/A}\}, \quad {}^J\mathbf{q}_B^A := {}^J\mathbf{q}\{-\phi_{BA}^{B/A}\}.} \tag{213}$$

For both conventions, the abstract quaternion $\underline{q} = q_0 + \underline{q}_v$ with quaternion coefficients $\{q_0, q_1, q_2, q_3\} \in \mathbb{R}$ represents the same rotation operation and the corresponding matrix realization can be computed

by (199) and (203) as:

$$\begin{aligned}
\mathbf{R}_B^A &= \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}, \\
\mathbf{R}_A^B &= \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 + q_0q_3) & 2(q_1q_3 - q_0q_2) \\ 2(q_1q_2 - q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 + q_0q_1) \\ 2(q_1q_3 + q_0q_2) & 2(q_2q_3 - q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}.
\end{aligned} \tag{214}$$

6.9 Quaternion Composition

Let F_A , F_B , and F_C be frames. The quaternions can be composed such that:

$$\boxed{\mathbf{q}_C^A = \mathbf{q}_B^A \otimes \mathbf{q}_C^B, \quad \mathbf{J}\mathbf{q}_C^A = \mathbf{J}\mathbf{q}_B^A \odot \mathbf{J}\mathbf{q}_C^B.} \tag{215}$$

To see this, let $\mathbf{H}\mathbf{x}^A, \mathbf{H}\mathbf{x}^B, \mathbf{H}\mathbf{x}^C \in \mathbb{R}^4$ be the Hamilton quaternion vector embeddings of component vectors of $\vec{\mathbf{x}} \in \mathbb{V}$ in F_A , F_B , and F_C , respectively. We have:

$$\begin{aligned}
\mathbf{H}\mathbf{x}^A &= \mathbf{H}\mathbf{q}_B^A \otimes \mathbf{x}^B \otimes (\mathbf{H}\mathbf{q}_B^A)^* \\
&= \mathbf{H}\mathbf{q}_B^A \otimes (\mathbf{H}\mathbf{q}_C^B \otimes \mathbf{H}\mathbf{x}^C \otimes (\mathbf{H}\mathbf{q}_C^B)^*) \otimes (\mathbf{H}\mathbf{q}_B^A)^* \\
&= \underbrace{(\mathbf{H}\mathbf{q}_B^A \otimes \mathbf{H}\mathbf{q}_C^B)}_{\mathbf{H}\mathbf{q}_C^A} \otimes \mathbf{H}\mathbf{x}^C \otimes (\mathbf{H}\mathbf{q}_B^A \otimes \mathbf{H}\mathbf{q}_C^B)^*.
\end{aligned} \tag{216}$$

6.10 Infinitesimal Rotations

In case of a small rotation $\delta\mathbf{q} \in \mathbb{H}$, we have:

$$\begin{aligned}
\delta\mathbf{q} &= \delta q_0 + \delta\mathbf{q}_v \\
&= \cos \frac{\delta\theta}{2} + \vec{u} \sin \frac{\delta\theta}{2}.
\end{aligned} \tag{217}$$

Let $\delta\phi_{BA}^{B/A} = \mathbf{u}^{B/A}\delta\theta \in \mathbb{R}^3$ be the associated rotation vector. Then, its quaternion vector representations can be computed as:

$$\begin{aligned}
\delta^{\mathbf{H}}\mathbf{q}_B^A &= \begin{bmatrix} \cos \frac{\delta\theta}{2} \\ \mathbf{u}^{B/A} \sin \frac{\delta\theta}{2} \end{bmatrix} \approx \begin{bmatrix} 1 \\ \frac{1}{2}\delta\phi_{BA}^{B/A} \end{bmatrix}, \\
\delta^{\mathbf{J}}\mathbf{q}_B^A &= \begin{bmatrix} -\mathbf{u}^{B/A} \sin \frac{\delta\theta}{2} \\ \cos \frac{\delta\theta}{2} \end{bmatrix} \approx \begin{bmatrix} -\frac{1}{2}\delta\phi_{BA}^{B/A} \\ 1 \end{bmatrix}.
\end{aligned} \tag{218}$$

7 Euler Angles

Euler angles describe a rotation as three successive rotations about principal axes. Therefore, any rotation matrix $\mathbf{R} \in SO(3)$ can be decomposed as a product of three principal rotation matrices

(Section 4.4) using the Euler angles. There are 12 valid Euler angle conventions, divided into two classes:

1. *Proper Euler Angles*: The first and third rotations are about the *same axis*. Hence, the axis sequences would be: 313, 323, 121, 131, 212, 232. Historically, proper Euler angles are used in rigid-body dynamics and orbital mechanics.
2. *Tait–Bryan Angles*: All three rotations are about different axes, hence the rotation sequences can be: 123, 132, 213, 231, 312, 321. All Tait–Bryan angles are Euler angles, but not all Euler angles are Tait–Bryan angles.

In this document, we will use the Tait–Bryan angle convention for Euler angles. As discussed in Section 3.5, Euler angles are a locally valid representation of $SO(3)$ and use three parameters to represent orientation: *roll* ϕ , *pitch* θ , and *yaw* ψ . The Euler angles vector $\boldsymbol{\psi}_{BA}$ encodes the orientation of frame F_B relative to frame F_A :

$$\boldsymbol{\Psi}_{BA} = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \in \mathbb{R}^3. \quad (219)$$

Note that $\boldsymbol{\Psi}_{BA}$ is not resolved in any frames since it is a parameterization of \mathbf{R}_B^A .

Euler angles suffer from *gimbal lock* — a singularity where two axes align and one degree of freedom is lost. The location of this singularity depends on the Euler angle convention used (e.g., at $\pm 90^\circ$ pitch for the common 321 aerospace sequence). However, Euler angles are the most intuitive representation for visualizing attitude.

7.1 Intrinsic and Extrinsic Rotations

Let F_A be a frame with basis vectors $\{\hat{\mathbf{i}}_A, \hat{\mathbf{j}}_A, \hat{\mathbf{k}}_A\}$. Let F_B be a frame obtained by applying three successive principal rotations to the basis vectors of F_A .

The three principal rotations may be *extrinsic* (rotations about the 1–, 2–, and 3–axis of the original frame, which is assumed to remain motionless), or *intrinsic* (rotations about the axes of the rotating frame, fixed to the moving body, which changes its orientation with respect to the extrinsic frame after each principal rotation).

In this document, we will discuss intrinsic rotations for the 321 and 123 Euler angle sequences.

7.2 321 Euler Sequence

7.2.1 321 Euler Angles to Rotation Matrix

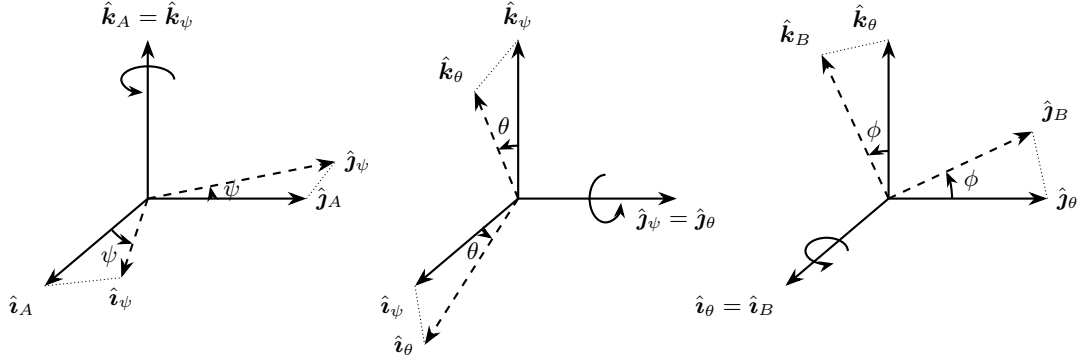


Figure 5: 321 Euler sequence from left to right

Let F_A and F_B be frames. Let $\psi_{BA} = [\phi, \theta, \psi]^T \in \mathbb{R}^3$ be the Euler angles vector. As illustrated in Figure 5, 321 intrinsic rotation is composed of:

1. Rotation of F_A basis vectors $\{\hat{i}_A, \hat{j}_A, \hat{k}_A\}$ through the yaw angle ψ about the 3-axis of F_A using the right-hand rule to obtain the first *intermediate frame* F_ψ :

$$F_\psi = F_A \mathbf{R}_\psi^A := F_A \mathbf{R}_3(\psi). \quad (220)$$

2. Rotation of F_ψ basis vectors $\{\hat{i}_\psi, \hat{j}_\psi, \hat{k}_\psi\}$ through the pitch angle θ about the 2-axis of F_ψ using the right-hand rule to obtain the second *intermediate frame* F_θ :

$$F_\theta = F_\psi \mathbf{R}_\theta^\psi := F_\psi \mathbf{R}_2(\theta). \quad (221)$$

3. Rotation of F_θ basis vectors $\{\hat{i}_\theta, \hat{j}_\theta, \hat{k}_\theta\}$ through the roll angle ϕ about the 1-axis of F_θ using the right-hand rule to obtain F_B :

$$F_B = F_\theta \mathbf{R}_\theta^\phi := F_\theta \mathbf{R}_1(\phi). \quad (222)$$

Therefore, we have:

$$\begin{aligned} F_B &= F_A \mathbf{R}_\psi^A \mathbf{R}_\theta^\psi \mathbf{R}_\theta^\phi \\ &= F_A \underbrace{(\mathbf{R}_3(\psi) \mathbf{R}_2(\theta) \mathbf{R}_1(\phi))}_{\mathbf{R}\{\psi_{BA}\}}, \end{aligned} \quad (223)$$

where we define the rotation matrix parameterized by the 321 Euler angles as:

$$\mathbf{R}\{\psi_{BA}\} := \mathbf{R}_3(\psi) \mathbf{R}_2(\theta) \mathbf{R}_1(\phi). \quad (224)$$

Using the above, the rotation matrix can be obtained as:

$$\begin{aligned}
\mathbf{R}_B^A &= \mathbf{R}\{\psi_{BA}\} \\
&= \mathbf{R}_3(\psi)\mathbf{R}_2(\theta)\mathbf{R}_1(\phi) \\
&= \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix} \\
&= \begin{bmatrix} c\psi c\theta & -s\psi c\phi + c\psi s\theta s\phi & s\psi s\phi + c\psi s\theta c\phi \\ s\psi c\theta & c\psi c\phi + s\psi s\theta s\phi & -c\psi s\phi + s\psi s\theta c\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix}.
\end{aligned} \tag{225}$$

Hence, we have:

$$\begin{aligned}
\mathbf{R}_B^A &= \begin{bmatrix} c\psi c\theta & -s\psi c\phi + c\psi s\theta s\phi & s\psi s\phi + c\psi s\theta c\phi \\ s\psi c\theta & c\psi c\phi + s\psi s\theta s\phi & -c\psi s\phi + s\psi s\theta c\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix}, \\
\mathbf{R}_A^B &= \begin{bmatrix} c\psi c\theta & s\psi c\theta & -s\theta \\ -s\psi c\phi + c\psi s\theta s\phi & c\psi c\phi + s\psi s\theta s\phi & c\theta s\phi \\ s\psi s\phi + c\psi s\theta c\phi & -c\psi s\phi + s\psi s\theta c\phi & c\theta c\phi \end{bmatrix}.
\end{aligned} \tag{226}$$

7.2.2 Rotation Matrix to 321 Euler Angles

Let R_{ij} denote the (i, j) entry of 321 rotation matrix \mathbf{R}_B^A . The Euler angles can be obtained from the rotation matrix as:

$$\begin{aligned}
\phi &= \arctan_2(R_{32}, R_{33}), \\
\theta &= -\arcsin(R_{31}), \\
\psi &= \arctan_2(R_{21}, R_{11}).
\end{aligned} \tag{227}$$

7.3 123 Euler Sequence

7.3.1 123 Euler Angles to Rotation Matrix

Let F_A and F_B be frames. Let $\psi_{BA} = [\phi, \theta, \psi]^T \in \mathbb{R}^3$ be the Euler angles vector. The 123 intrinsic rotation is composed of:

1. Rotation of F_A basis vectors $\{\hat{\mathbf{i}}_A, \hat{\mathbf{j}}_A, \hat{\mathbf{k}}_A\}$ through the roll angle ϕ about the 1-axis of F_A using the right-hand rule to obtain the first *intermediate frame* F_ϕ :

$$F_\phi = F_A \mathbf{R}_\phi^A := F_A \mathbf{R}_1(\phi). \tag{228}$$

2. Rotation of F_ϕ basis vectors $\{\hat{\mathbf{i}}_\phi, \hat{\mathbf{j}}_\phi, \hat{\mathbf{k}}_\phi\}$ through the pitch angle θ about the 2-axis of F_ϕ using the right-hand rule to obtain the second *intermediate frame* F_θ :

$$F_\theta = F_\phi \mathbf{R}_\theta^\phi := F_\phi \mathbf{R}_2(\theta). \tag{229}$$

3. Rotation of F_θ basis vectors $\{\hat{\mathbf{i}}_\theta, \hat{\mathbf{j}}_\theta, \hat{\mathbf{k}}_\theta\}$ through the yaw angle ψ about the 3-axis of F_θ using the right-hand rule to obtain F_B :

$$F_B = F_\theta \mathbf{R}_B^\theta := F_\theta \mathbf{R}_3(\psi). \quad (230)$$

Therefore, we have:

$$\begin{aligned} F_B &= F_A \mathbf{R}_\phi^A \mathbf{R}_\theta^\phi \mathbf{R}_B^\theta \\ &= F_A \underbrace{(\mathbf{R}_1(\phi) \mathbf{R}_2(\theta) \mathbf{R}_3(\psi))}_{\mathbf{R}\{\psi_{BA}\}}, \end{aligned} \quad (231)$$

where we define the rotation matrix parameterized by the 123 Euler angles as:

$$\mathbf{R}\{\psi_{BA}\} := \mathbf{R}_1(\phi) \mathbf{R}_2(\theta) \mathbf{R}_3(\psi). \quad (232)$$

Using the above, the rotation matrix can be obtained as:

$$\begin{aligned} \mathbf{R}_B^A &= \mathbf{R}\{\Psi_{BA}\} \\ &= \mathbf{R}_1(\phi) \mathbf{R}_2(\theta) \mathbf{R}_3(\psi) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix} \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c\theta c\psi & -c\theta s\psi & s\theta \\ c\phi s\psi + s\phi s\theta c\psi & c\phi c\psi - s\phi s\theta s\psi & -s\phi c\theta \\ s\phi s\psi - c\phi s\theta c\psi & s\phi c\psi + c\phi s\theta s\psi & c\phi c\theta \end{bmatrix}. \end{aligned} \quad (233)$$

Hence, we have:

$$\begin{aligned} \mathbf{R}_B^A &= \begin{bmatrix} c\theta c\psi & -c\theta s\psi & s\theta \\ c\phi s\psi + s\phi s\theta c\psi & c\phi c\psi - s\phi s\theta s\psi & -s\phi c\theta \\ s\phi s\psi - c\phi s\theta c\psi & s\phi c\psi + c\phi s\theta s\psi & c\phi c\theta \end{bmatrix}, \\ \mathbf{R}_A^B &= \begin{bmatrix} c\theta c\psi & c\phi s\psi + s\phi s\theta c\psi & s\phi s\psi - c\phi s\theta c\psi \\ -c\theta s\psi & c\phi c\psi - s\phi s\theta s\psi & s\phi c\psi + c\phi s\theta s\psi \\ s\theta & -s\phi c\theta & c\phi c\theta \end{bmatrix}. \end{aligned} \quad (234)$$

7.3.2 Rotation Matrix to 123 Euler Angles

Let R_{ij} denote the (i, j) entry of the 123 rotation matrix \mathbf{R}_B^A . The Euler angles can be obtained from the rotation matrix as:

$$\begin{aligned} \phi &= \arctan_2(-R_{23}, R_{33}), \\ \theta &= \arcsin(R_{13}), \\ \psi &= \arctan_2(-R_{12}, R_{11}). \end{aligned} \quad (235)$$

7.4 Infinitesimal Rotations

Consider 321 and 123 rotations when the Euler angles $\delta\psi$, $\delta\theta$, and $\delta\phi$ are small. For each Euler angle, we can make the following approximations:

$$\cos \delta\theta \approx 1, \quad \sin \delta\theta \approx \delta\theta, \quad \delta\theta\delta\phi = \delta\theta\delta\psi = 0. \quad (236)$$

Rotation matrices of 321 and 123 sequences can be both approximated to first order as:

$$\mathbf{R}_B^A \approx \begin{bmatrix} 1 & -\delta\psi & \delta\theta \\ \delta\psi & 1 & -\delta\phi \\ -\delta\theta & \delta\phi & 1 \end{bmatrix}, \quad \mathbf{R}_A^B \approx \begin{bmatrix} 1 & \delta\psi & -\delta\theta \\ -\delta\psi & 1 & \delta\phi \\ \delta\theta & -\delta\phi & 1 \end{bmatrix}. \quad (237)$$

Therefore, we get:

$$\boxed{\begin{aligned} \mathbf{R}_B^A &\approx \mathbf{I} + [\delta\psi_{BA}]_{\times}, \\ \mathbf{R}_A^B &\approx \mathbf{I} - [\delta\psi_{BA}]_{\times}, \end{aligned}} \quad (238)$$

where $\delta\psi_{BA} = [\delta\phi, \delta\theta, \delta\psi]^T$.

IV Time-Varying Rotations

In most kinematics applications, rotations change with time. Therefore, in this part, we discuss the time derivatives of rotation matrices, quaternions, and Euler angles. The rate of change is described by the angular velocity and angular acceleration.

8 Rotating Frames

8.1 Angular Velocity

Let F_A and F_B be frames with basis vectors $\{\hat{i}_A, \hat{j}_A, \hat{k}_A\}$ and $\{\hat{i}_B, \hat{j}_B, \hat{k}_B\}$, respectively. The *angular velocity of the frame F_B relative to the frame F_A* is denoted as $\vec{\omega}_{BA} \in \mathbb{V}$. Conversely, the angular velocity of frame F_A with respect to F_B is $\vec{\omega}_{AB} = -\vec{\omega}_{BA}$.

The magnitude of $\vec{\omega}_{BA}$, $\|\vec{\omega}_{BA}\|$, is the rate of rotation, and the unit vector direction $\vec{\omega}_{BA}/\|\vec{\omega}_{BA}\|$ is the *instantaneous* axis of rotation.

The physical angular velocity vector $\vec{\omega}_{BA}$ can be resolved in frame F_C as:

$$[\vec{\omega}_{BA}]_C \equiv \omega_{BA}^C \in \mathbb{R}^3. \quad (239)$$

Given a frame F_D , the resolving axes may be changed by premultiplying with the rotation matrix $\mathbf{R}_C^D \in SO(3)$:

$$\omega_{BA}^D = \mathbf{R}_C^D \omega_{BA}^C. \quad (240)$$

Angular velocities resolved about the same frame axes may simply be added:

$$\omega_{BA}^C = \omega_{BD}^C + \omega_{DA}^C. \quad (241)$$

The skew-symmetric matrix (i.e., equal to the negative of its transpose) of the component angular velocity vector is defined as:

$$\Omega_{BA}^C := [\omega_{BA}^C]_{\times} \in \mathbb{R}^{3 \times 3}. \quad (242)$$

The resolving frame can be changed as:

$$\Omega_{BA}^D = [\omega_{BA}^D]_{\times} \quad (243)$$

$$= [\mathbf{R}_C^D \omega_{BA}^C]_{\times} \quad (244)$$

$$= \mathbf{R}_C^D [\omega_{BA}^C]_{\times} (\mathbf{R}_C^D)^T \quad [\mathbf{R}\mathbf{x}]_{\times} = \mathbf{R} [\mathbf{x}]_{\times} \mathbf{R}^T. \quad (245)$$

$$= \mathbf{R}_C^D \Omega_{BA}^C (\mathbf{R}_C^D)^T \quad (246)$$

$$= \mathbf{R}_C^D \Omega_{BA}^C \mathbf{R}_D^C. \quad (247)$$

8.2 Rotation Vector and Angular Velocity Relationship

When $\vec{\omega}_{BA}$ is constant over an interval $[0, t]$, the rotation accumulated over that interval is a rotation by angle $\|\omega_{BA}^B\|t$ about the fixed axis $\omega_{BA}^B/\|\omega_{BA}^B\|$. The accumulated rotation can be expressed using a rotation vector from Section 5:

$$\phi_{BA}^B(t) = \omega_{BA}^B t \in \mathbb{R}^3. \quad (248)$$

If $\vec{\omega}_{BA}(t)$ is time-varying, for an infinitesimal time step $\delta t > 0$, the angular velocity can be treated as constant to first order. The rotation accumulated over $[t, t + \delta t]$ is then encoded by the infinitesimal rotation vector:

$$\delta\phi_{BA}^B := \omega_{BA}^B \delta t \in \mathbb{R}^3, \quad (249)$$

and the corresponding rotation matrix admits the first-order expansion:

$$\exp\left([\omega_{BA}^B \delta t]_{\times}\right) = \mathbf{I} + [\omega_{BA}^B]_{\times} \delta t + \mathcal{O}(\delta t^2). \quad (250)$$

Over a short time window, ω_{BA}^B is the rate at which the rotation vector accumulates. This is the sense in which the angular velocity *generates* the rotation. In more general case, integrating the angular velocity over $[0, t]$ does not yield the rotation vector:

$$\phi_{BA}^B(t) \neq \int_0^t \omega_{BA}^B(\tau) d\tau. \quad (251)$$

Because $SO(3)$ is non-commutative, the infinitesimal rotations generated by $\omega_{BA}^B(\tau)$ at different times do not compose by addition. The correct relationship between the time derivative of the rotation vector and the angular velocity involves the *right Jacobian* of $SO(3)$, $\mathbf{J}_r(\phi) \in \mathbb{R}^{3 \times 3}$:

$$\omega_{BA}^B = \mathbf{J}_r(\phi_{BA}^B) \dot{\phi}_{BA}^B, \quad \dot{\phi}_{BA}^B = \mathbf{J}_r^{-1}(\phi_{BA}^B) \omega_{BA}^B. \quad (252)$$

At $\phi = \mathbf{0}$, the right Jacobian reduces to the identity, $\mathbf{J}_r(\mathbf{0}) = \mathbf{I}$, which recovers (249) in the infinitesimal limit. For finite ϕ , $\mathbf{J}_r(\phi) \neq \mathbf{I}$, and $\dot{\phi}$ differs from ω by a correction that accounts for the curvature of $SO(3)$. A derivation of \mathbf{J}_r and its inverse is deferred to Section 13.

8.3 Vectrix Derivatives

The vectrix derivative of F_B with respect to F_A is defined as (refer to Section 1.3):

$$\hat{\underset{\cdot}{F}}_B^A := \begin{bmatrix} \hat{\underset{\cdot}{i}}_B^A & \hat{\underset{\cdot}{j}}_B^A & \hat{\underset{\cdot}{k}}_B^A \end{bmatrix} \in \mathbb{V}^{3 \times 1}. \quad (253)$$

To compute each term in (253), we will use the normality and orthogonality constraints of basis vectors.

Normality Constraints Recall that the basis vectors must satisfy the normality constraints:

$$\hat{i}_B \cdot \hat{i}_B = \hat{j}_B \cdot \hat{j}_B = \hat{k}_B \cdot \hat{k}_B = 1. \quad (254)$$

Differentiating the normality constraint with respect to frame F_A gives:

$$\hat{i}_B \cdot \hat{\underset{\cdot}{i}}_B^A = 0, \quad \hat{i}_B \cdot \hat{\underset{\cdot}{j}}_B^A = 0, \quad \hat{i}_B \cdot \hat{\underset{\cdot}{k}}_B^A = 0, \quad (255)$$

which implies:

$$\hat{\underset{\cdot}{i}}_B^A \perp \hat{i}_B, \quad \hat{\underset{\cdot}{j}}_B^A \perp \hat{j}_B, \quad \hat{\underset{\cdot}{k}}_B^A \perp \hat{k}_B. \quad (256)$$

Therefore, there exists vectors $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}} \in \mathbb{V}$ such that:

$$\hat{\underset{\cdot}{i}}_B^A = \vec{\mathbf{a}} \times \hat{i}_B, \quad \hat{\underset{\cdot}{j}}_B^A = \vec{\mathbf{b}} \times \hat{j}_B, \quad \hat{\underset{\cdot}{k}}_B^A = \vec{\mathbf{c}} \times \hat{k}_B. \quad (257)$$

The physical vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{c}}$ can be expressed by their component vectors as:

$$\vec{\mathbf{a}} = F_B \mathbf{a}^B = F_B \begin{bmatrix} 0 & a_2 & a_3 \end{bmatrix}^T, \quad (258)$$

$$\vec{\mathbf{b}} = F_B \mathbf{b}^B = F_B \begin{bmatrix} b_1 & 0 & b_3 \end{bmatrix}^T, \quad (259)$$

$$\vec{\mathbf{c}} = F_B \mathbf{c}^B = F_B \begin{bmatrix} c_1 & c_2 & 0 \end{bmatrix}^T, \quad (260)$$

since each physical vector is perpendicular to single basis vector of frame F_B .

Orthogonality Constraints Recall that the basis vectors must satisfy the orthogonality constraints:

$$\hat{\mathbf{i}}_B \cdot \hat{\mathbf{j}}_B = \hat{\mathbf{i}}_B \cdot \hat{\mathbf{k}}_B = \hat{\mathbf{j}}_B \cdot \hat{\mathbf{k}}_B = 0. \quad (261)$$

Differentiating the orthogonality constraints in frame F_A gives:

$$\overset{A}{\hat{\mathbf{i}}_B} \cdot \hat{\mathbf{j}}_B + \hat{\mathbf{i}}_B \cdot \overset{A}{\hat{\mathbf{j}}_B} = 0, \quad (262)$$

$$\overset{A}{\hat{\mathbf{i}}_B} \cdot \hat{\mathbf{k}}_B + \hat{\mathbf{i}}_B \cdot \overset{A}{\hat{\mathbf{k}}_B} = 0, \quad (263)$$

$$\overset{A}{\hat{\mathbf{j}}_B} \cdot \hat{\mathbf{k}}_B + \hat{\mathbf{j}}_B \cdot \overset{A}{\hat{\mathbf{k}}_B} = 0. \quad (264)$$

Substituting (257) into above gives:

$$(\vec{\mathbf{a}} \times \hat{\mathbf{i}}_B) \cdot \hat{\mathbf{j}}_B + \hat{\mathbf{i}}_B \cdot (\vec{\mathbf{b}} \times \hat{\mathbf{j}}_B) = 0, \quad (265)$$

$$(\vec{\mathbf{a}} \times \hat{\mathbf{i}}_B) \cdot \hat{\mathbf{k}}_B + \hat{\mathbf{i}}_B \cdot (\vec{\mathbf{c}} \times \hat{\mathbf{k}}_B) = 0, \quad (266)$$

$$(\vec{\mathbf{b}} \times \hat{\mathbf{j}}_B) \cdot \hat{\mathbf{k}}_B + \hat{\mathbf{j}}_B \cdot (\vec{\mathbf{c}} \times \hat{\mathbf{k}}_B) = 0. \quad (267)$$

Using the scalar triple product identity, we get:

$$(\vec{\mathbf{b}} - \vec{\mathbf{a}}) \cdot \hat{\mathbf{k}}_B = 0, \quad (268)$$

$$(\vec{\mathbf{c}} - \vec{\mathbf{a}}) \cdot \hat{\mathbf{k}}_B = 0, \quad (269)$$

$$(\vec{\mathbf{c}} - \vec{\mathbf{b}}) \cdot \hat{\mathbf{i}}_B = 0, \quad (270)$$

which leads to additional constraints:

$$b_3 = a_3, \quad c_1 = b_1, \quad c_2 = a_2. \quad (271)$$

Hence, the component vectors of $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$ must satisfy:

$$\vec{\mathbf{a}} = F_B \mathbf{a}^B = F_B \begin{bmatrix} 0 & a_2 & a_3 \end{bmatrix}^T, \quad (272)$$

$$\vec{\mathbf{b}} = F_B \mathbf{b}^B = F_B \begin{bmatrix} b_1 & 0 & a_3 \end{bmatrix}^T, \quad (273)$$

$$\vec{\mathbf{c}} = F_B \mathbf{c}^B = F_B \begin{bmatrix} b_1 & a_2 & 0 \end{bmatrix}^T. \quad (274)$$

Note that the physical vector $F_B \begin{bmatrix} b_1 & 0 & 0 \end{bmatrix}^T$ is parallel to the basis vector $\hat{\mathbf{i}}_B$, the physical vector $F_B \begin{bmatrix} 0 & a_2 & 0 \end{bmatrix}^T$ is parallel to the basis vector $\hat{\mathbf{j}}_B$, and the physical vector $F_B \begin{bmatrix} 0 & 0 & a_3 \end{bmatrix}^T$ is parallel

to the basis vector $\hat{\mathbf{k}}_B$. Let $\vec{\omega}_{BA} \in \mathbb{V}$ be the angular velocity vector of frame F_B relative to frame F_A such that:

$$\vec{\omega}_{BA} \hat{=} \vec{\mathbf{a}} + F_B \begin{bmatrix} b_1 & 0 & 0 \end{bmatrix}^T \quad (275)$$

$$= \vec{\mathbf{b}} + F_B \begin{bmatrix} 0 & a_2 & 0 \end{bmatrix}^T \quad (276)$$

$$= \vec{\mathbf{c}} + F_B \begin{bmatrix} 0 & 0 & a_3 \end{bmatrix}^T \quad (277)$$

$$= F_B \begin{bmatrix} b_1 & a_2 & a_3 \end{bmatrix}^T \quad (278)$$

$$= F_B \underbrace{\begin{bmatrix} {}^B\omega_1 & {}^B\omega_2 & {}^B\omega_3 \end{bmatrix}^T}_{{}^B\omega_{BA}}. \quad (279)$$

Using above, we have:

$$\hat{\mathbf{i}}_B^A = \vec{\mathbf{a}} \times \hat{\mathbf{i}}_B = \vec{\omega}_{BA} \times \hat{\mathbf{i}}_B, \quad (280)$$

$$\hat{\mathbf{j}}_B^A = \vec{\mathbf{b}} \times \hat{\mathbf{j}}_B = \vec{\omega}_{BA} \times \hat{\mathbf{j}}_B, \quad (281)$$

$$\hat{\mathbf{k}}_B^A = \vec{\mathbf{c}} \times \hat{\mathbf{k}}_B = \vec{\omega}_{BA} \times \hat{\mathbf{k}}_B. \quad (282)$$

Finally, (253) can be rewritten as:

$$\boxed{\begin{aligned} \hat{F}_B^A &= \begin{bmatrix} \hat{\mathbf{i}}_B^A & \hat{\mathbf{j}}_B^A & \hat{\mathbf{k}}_B^A \end{bmatrix} \\ &= \begin{bmatrix} \vec{\omega}_{BA} \times \hat{\mathbf{i}}_B & \vec{\omega}_{BA} \times \hat{\mathbf{j}}_B & \vec{\omega}_{BA} \times \hat{\mathbf{k}}_B \end{bmatrix} \\ &= \vec{\omega}_{BA} \times F_B = -(F_B \times \vec{\omega}_{BA}). \end{aligned}} \quad (283)$$

Similarly:

$$\begin{aligned} \hat{F}_A^B &= \begin{bmatrix} \hat{\mathbf{i}}_A^B & \hat{\mathbf{j}}_A^B & \hat{\mathbf{k}}_A^B \end{bmatrix} \\ &= \begin{bmatrix} \vec{\omega}_{AB} \times \hat{\mathbf{i}}_A & \vec{\omega}_{AB} \times \hat{\mathbf{j}}_A & \vec{\omega}_{AB} \times \hat{\mathbf{k}}_A \end{bmatrix} \\ &= \begin{bmatrix} -\vec{\omega}_{BA} \times \hat{\mathbf{i}}_A & -\vec{\omega}_{BA} \times \hat{\mathbf{j}}_A & -\vec{\omega}_{BA} \times \hat{\mathbf{k}}_A \end{bmatrix} \\ &= -\vec{\omega}_{BA} \times F_A = F_A \times \vec{\omega}_{BA}. \end{aligned} \quad (284)$$

9 Time Derivative of Rotations

9.1 Time Derivative of Rotation Matrices

Since $SO(3)$ is not a vector space, it is not closed under the usual vector space operations of addition and subtraction. Consequently, the standard limit definition of a derivative cannot be directly applied to a trajectory defined by $\mathbf{R}(t) \in SO(3)$ in the same way as for vectors in \mathbb{R}^n . To see how rotations evolve with time, we will derive the time derivative structure of rotation matrices using the vectrix derivatives.

Let F_A be a frame. Let $F_B(t)$ be a time-varying frame undergoing a continuous rotational motion with constant angular velocity vector $\vec{\omega}_{BA} \in \mathbb{V}$.

The orientation of F_B at time $t \in \mathbb{R}^+$ can be represented by $\mathbf{R}_{B(t)}^A \in SO(3)$, which is the rotation matrix from frame $F_B(t)$ to frame F_A . From (88), we have:

$$\mathbf{R}_{B(t)}^A = [F_B(t)]_A. \quad (285)$$

The *time derivative of $\mathbf{R}_{B(t)}^A$ with respect to frame F_A* can be defined as:

$$\dot{\mathbf{R}}_{B(t)}^A = \frac{d}{dt} ([F_B(t)]_A) \quad (286)$$

$$= \widehat{[F_B]_A}^A \quad (287)$$

$$= [\vec{\omega}_{BA} \times F_B(t)]_A \quad (288)$$

$$= [\vec{\omega}_{BA}]_A \times [F_B(t)]_A \quad (289)$$

$$= \omega_{BA}^A \times \mathbf{R}_{B(t)}^A \quad (290)$$

$$= -[\omega_{AB}^A]_{\times} \mathbf{R}_{B(t)}^A \quad (291)$$

$$= -\Omega_{AB}^A \mathbf{R}_{B(t)}^A, \quad (292)$$

where $\Omega_{AB}^A := [\omega_{AB}^A]_{\times} \in \mathbb{R}^{3 \times 3}$ is a skew-symmetric matrix (i.e., it is equal to the negative of its transpose).

Similarly, observer in F_B will see a time-varying frame $F_A(t)$ rotating with angular velocity angular velocity vector $\vec{\omega}_{AB} = -\vec{\omega}_{BA} \in \mathbb{V}$. Therefore, the time derivative of $\mathbf{R}_{A(t)}^B$ with respect to frame F_B can be computed as:

$$\dot{\mathbf{R}}_{A(t)}^B = \frac{d}{dt} ([F_A(t)]_B) \quad (293)$$

$$= \widehat{[F_A]_B}^B$$

$$= [-\vec{\omega}_{BA} \times F_A(t)]_B$$

$$= [-\vec{\omega}_{BA}]_B \times [F_A(t)]_B$$

$$= -[\omega_{BA}^B]_{\times} \mathbf{R}_{A(t)}^B$$

$$= -\Omega_{BA}^B \mathbf{R}_{A(t)}^B.$$

Since $\Omega_{BA}^A = -\Omega_{AB}^A$ and $\Omega_{AB}^B = -\Omega_{BA}^B$, we have the following properties:

$$\boxed{\begin{aligned} \dot{\mathbf{R}}_{B(t)}^A &= -\Omega_{AB}^A \mathbf{R}_{B(t)}^A = \Omega_{BA}^A \mathbf{R}_{B(t)}^A, \\ \dot{\mathbf{R}}_{A(t)}^B &= -\Omega_{BA}^B \mathbf{R}_{A(t)}^B = \Omega_{AB}^B \mathbf{R}_{A(t)}^B. \end{aligned}} \quad (294)$$

From (247), we can get additional properties as:

$$\boxed{\begin{aligned} \dot{\mathbf{R}}_{B(t)}^A &= \mathbf{R}_{B(t)}^A \Omega_{BA}^B = -\mathbf{R}_{B(t)}^A \Omega_{AB}^B, \\ \dot{\mathbf{R}}_{A(t)}^B &= \mathbf{R}_{A(t)}^B \Omega_{AB}^A = -\mathbf{R}_{A(t)}^B \Omega_{BA}^A. \end{aligned}} \quad (295)$$

The closed-form solution to (295) is given by:

$$\boxed{\mathbf{R}_{B(t)}^A = \mathbf{R}_{B(0)}^A \exp(\Omega_{BA}^B t) = \mathbf{R}_{B(0)}^A \exp([\omega_{BA}^B t]_{\times})}, \quad (296)$$

where $\mathbf{R}_{B(0)}^A$ is the initial orientation of F_B relative to F_A at $t = 0$. The full derivation is provided in Section ??.

9.2 Time Derivative of Quaternions

Let $\underline{q}_{B(t)}^A \in S^3$ be a time-varying abstract unit quaternion and let $\mathbf{q}_{B(t)}^A := {}^H\mathbf{q}_{B(t)}^A \in \mathbb{R}^4$ be its Hamilton quaternion vector form representing the orientation of $F_B(t)$ relative to F_A . Let $\boldsymbol{\omega}_{BA}^B \in \mathbb{R}^3$ be the angular velocity of F_B relative to F_A resolved in F_B . Since quaternion sum is well defined in Section 6.3.1, we can compute the time derivative by computing the limit of the difference quotient:

$$\dot{\mathbf{q}}_{B(t)}^A := \lim_{\delta t \rightarrow 0} \frac{\mathbf{q}_{B(t+\delta t)}^A - \mathbf{q}_{B(t)}^A}{\delta t}. \quad (297)$$

The quaternion $\mathbf{q}_{B(t+\delta t)}^A$ can be expressed as the composition of two quaternions:

$$\mathbf{q}_{B(t+\delta t)}^A = \mathbf{q}_{B(t)}^A \otimes \mathbf{q}_{B(t+\delta t)}^{B(t)}, \quad (298)$$

where $\mathbf{q}_{B(t+\delta t)}^{B(t)}$ describes the rotation of frame $F_B(t + \delta t)$ to frame $F_B(t)$. For an infinitesimal time step $\delta t > 0$, the rotation accumulated over $[t, t + \delta t]$ is encoded by the infinitesimal rotation vector $\delta\boldsymbol{\phi}_{BA}^B := \boldsymbol{\omega}_{BA}^B \delta t$ (Section 8.2) and therefore:

$$\mathbf{q}_{B(t+\delta t)}^{B(t)} = \begin{bmatrix} \cos \frac{\theta}{2} \\ \mathbf{u}^B \sin \frac{\theta}{2} \end{bmatrix} \approx \begin{bmatrix} 1 \\ \frac{1}{2}\delta\boldsymbol{\phi}_{BA}^B \end{bmatrix}. \quad (299)$$

Using above, (297) can be written as:

$$\begin{aligned} \dot{\mathbf{q}}_{B(t)}^A &= \lim_{\delta t \rightarrow 0} \frac{\mathbf{q}_{B(t+\delta t)}^A - \mathbf{q}_{B(t)}^A}{\delta t} \\ &= \lim_{\delta t \rightarrow 0} \frac{\mathbf{q}_{B(t)}^A \otimes \mathbf{q}_{B(t+\delta t)}^{B(t)} - \mathbf{q}_{B(t)}^A \otimes \mathbf{q}_0}{\delta t} \\ &= \lim_{\delta t \rightarrow 0} \frac{\mathbf{q}_{B(t)}^A \otimes (\mathbf{q}_{B(t+\delta t)}^{B(t)} - \mathbf{q}_0)}{\delta t} \\ &\approx \lim_{\delta t \rightarrow 0} \frac{\mathbf{q}_{B(t)}^A \otimes \left(\begin{bmatrix} 1 \\ \frac{1}{2}\delta\boldsymbol{\phi}_{BA}^B \end{bmatrix} - \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} \right)}{\delta t} \\ &= \mathbf{q}_{B(t)}^A \otimes \begin{bmatrix} 1 \\ \frac{1}{2}\delta\boldsymbol{\omega}_{BA}^B \end{bmatrix} \\ &= \frac{1}{2}\mathbf{q}_{B(t)}^A \otimes {}^H\boldsymbol{\omega}_{BA}^B. \end{aligned} \quad (300)$$

Hence, we have:

$$\boxed{\dot{\mathbf{q}}_{B(t)}^A = \frac{1}{2}\mathbf{q}_{B(t)}^A \otimes {}^H\boldsymbol{\omega}_{BA}^B.} \quad (301)$$

This is the quaternion analogue of the rotation matrix ODE $\dot{\mathbf{R}}_{B(t)}^A = \mathbf{R}_{B(t)}^A \boldsymbol{\Omega}_{BA}^B$ derived in (295): the angular velocity is right-multiplied and resolved in the source frame F_B . The factor $\frac{1}{2}$ arises from the double-cover $S^3 \rightarrow SO(3)$ (Section 3.2.2): a physical rotation by angle θ corresponds to a quaternion with $\sin(\theta/2)$ in its vector part, so infinitesimal angular velocity enters with a factor of $1/2$.

Using the Hamiltonian right product matrix $\mathbf{q} \otimes \mathbf{p} = [\mathbf{p}]_R \mathbf{q}$ (Table 5), the ODE (301) can be written

as:

$$\dot{\mathbf{q}}_{B(t)}^A = \frac{1}{2} [{}^H\boldsymbol{\omega}_{BA}^B]_R \mathbf{q}_{B(t)}^A = \frac{1}{2} \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix} \mathbf{q}_{B(t)}^A. \quad (302)$$

Similarly, $\dot{\mathbf{q}}_{A(t)}^B$ can be obtained by considering rotation with the reverse angular velocity vector $\boldsymbol{\phi}_{AB}^A = -\boldsymbol{\phi}_{BA}^A$:

$$\dot{\mathbf{q}}_{A(t)}^B = -\frac{1}{2} \mathbf{q}_{A(t)}^B \otimes {}^H\boldsymbol{\omega}_{BA}^A. \quad (303)$$

9.3 Time Derivative of Euler Angles

9.3.1 Time Derivative of Principal Rotation Matrices

Before we compute the time derivative of Euler angles (or *Euler rates*), it is useful to compute the derivatives of principal rotation matrices defined in Section 4.4. Let the angles of rotations about 1-, 2-, and 3- axis be $\varphi(t)$, $\theta(t)$, and $\psi(t)$, respectively. Let $\mathbf{i}_j \in \mathbb{R}^3$ be a unit column vector whose entries are all zero except the j -th entry, which is 1. From (295), we have:

$$\begin{aligned} \dot{\mathbf{R}}_1(\varphi) &= \mathbf{R}_1(\varphi)[\mathbf{i}_1]_{\times} \dot{\varphi}, \\ \dot{\mathbf{R}}_2(\theta) &= \mathbf{R}_2(\theta)[\mathbf{i}_2]_{\times} \dot{\theta}, \\ \dot{\mathbf{R}}_3(\psi) &= \mathbf{R}_3(\psi)[\mathbf{i}_3]_{\times} \dot{\psi}, \end{aligned} \quad (304)$$

or equivalently:

$$\begin{aligned} [\mathbf{i}_1]_{\times} \dot{\varphi} &= \mathbf{R}_1^T(\varphi) \dot{\mathbf{R}}_1(\varphi), \\ [\mathbf{i}_2]_{\times} \dot{\theta} &= \mathbf{R}_2^T(\theta) \dot{\mathbf{R}}_2(\theta), \\ [\mathbf{i}_3]_{\times} \dot{\psi} &= \mathbf{R}_3^T(\psi) \dot{\mathbf{R}}_3(\psi). \end{aligned} \quad (305)$$

9.3.2 321 Euler Rates

Let F_A be a frame and let F_B be a time-varying frame. Let $\boldsymbol{\Psi}_{BA}(t) = [\varphi(t), \theta(t), \psi(t)]^T \in \mathbb{R}^3$ be the Euler angles vector describing the orientation of $F_B(t)$ relative to F_A . The rotation matrix obtained from 321 Euler angle sequence is defined by (225):

$$\mathbf{R}_{B(t)}^A = \mathbf{R}\{\boldsymbol{\Psi}_{BA}(t)\} = \mathbf{R}_3(\psi(t))\mathbf{R}_2(\theta(t))\mathbf{R}_1(\varphi(t)). \quad (306)$$

From (295) and (305), we get:

$$\begin{aligned} \boldsymbol{\Omega}_{BA}^B &= \left(\mathbf{R}_{B(t)}^A\right)^T \dot{\mathbf{R}}_{B(t)}^A \\ &= (\mathbf{R}_3\mathbf{R}_2\mathbf{R}_1)^T \overbrace{\dot{\mathbf{R}}_3\mathbf{R}_2\mathbf{R}_1} \\ &= \mathbf{R}_1^T\mathbf{R}_2^T\mathbf{R}_3^T \left[\dot{\mathbf{R}}_3\mathbf{R}_2\mathbf{R}_1 + \mathbf{R}_3\dot{\mathbf{R}}_2\mathbf{R}_1 + \mathbf{R}_3\mathbf{R}_2\dot{\mathbf{R}}_1 \right] \\ &= \mathbf{R}_1^T\mathbf{R}_2^T\mathbf{R}_3^T \dot{\mathbf{R}}_3\mathbf{R}_2\mathbf{R}_1 + \mathbf{R}_1^T\mathbf{R}_2^T\dot{\mathbf{R}}_2\mathbf{R}_1 + \mathbf{R}_1^T\dot{\mathbf{R}}_1 \\ &= \mathbf{R}_1^T\mathbf{R}_2^T[\mathbf{i}_3]_{\times} \dot{\psi}\mathbf{R}_2\mathbf{R}_1 + \mathbf{R}_1^T[\mathbf{i}_2]_{\times} \dot{\theta}\mathbf{R}_1 + [\mathbf{i}_1]_{\times} \dot{\varphi} \\ &= \left[\mathbf{R}_1^T\mathbf{R}_2^T\mathbf{i}_3\dot{\psi} \right]_{\times} + \left[\mathbf{R}_1^T\mathbf{i}_2\dot{\theta} \right]_{\times} + [\mathbf{i}_1\dot{\varphi}]_{\times}. \end{aligned} \quad (307)$$

Therefore, we have:

$$\omega_{BA}^B = \begin{bmatrix} \mathbf{i}_1 & \mathbf{R}_1^T \mathbf{i}_2 & \mathbf{R}_1^T \mathbf{R}_2^T \mathbf{i}_3 \end{bmatrix} \begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \mathbf{S}_{321}(\varphi(t), \theta(t)) \dot{\Psi}_{BA}, \quad (308)$$

where we defined $\mathbf{S}_{321}(\varphi, \theta) = [\mathbf{i}_1, \mathbf{R}_1^T \mathbf{i}_2, \mathbf{R}_1^T \mathbf{R}_2^T \mathbf{i}_3] \in \mathbb{R}^{3 \times 3}$. Equation (308) relates the angular velocity in terms of the Euler angles and the Euler rates. We have:

$$\mathbf{S}_{321}(\varphi(t), \theta(t)) = \begin{bmatrix} 1 & 0 & -\sin \theta(t) \\ 0 & \cos \varphi(t) & \sin \varphi(t) \cos \theta(t) \\ 0 & -\sin \varphi(t) & \cos \varphi(t) \cos \theta(t) \end{bmatrix}. \quad (309)$$

The inverse relationship can be computed by finding the inverse \mathbf{S}_{321}^{-1} :

$$\begin{aligned} \dot{\Psi}_{BA} &= \mathbf{S}_{321}^{-1}(\varphi, \theta) \omega_{BA}^B \\ &= \begin{bmatrix} 1 & \sin \varphi(t) \tan \theta(t) & \cos \varphi(t) \tan \theta(t) \\ 0 & \cos \varphi(t) & -\sin \varphi(t) \\ 0 & \sin \varphi(t) / \cos \theta(t) & \cos \varphi(t) / \cos \theta(t) \end{bmatrix} \omega_{BA}^B. \end{aligned} \quad (310)$$

Note that \mathbf{S}_{321}^{-1} does not exist at $\theta = \pi/2$.

9.3.3 123 Euler Rates

Let F_A be a frame and let F_B be a time-varying frame. Let $\Psi_{BA}(t) = [\varphi(t), \theta(t), \psi(t)]^T \in \mathbb{R}^3$ be the Euler angles vector describing the orientation of $F_B(t)$ relative to F_A . The rotation matrix obtained from 123 Euler angle sequence is defined by (233):

$$\mathbf{R}_{B(t)}^A = \mathbf{R}\{\Psi_{BA}(t)\} = \mathbf{R}_1(\phi(t)) \mathbf{R}_2(\theta(t)) \mathbf{R}_3(\psi(t)). \quad (311)$$

From (295) and (305), we get:

$$\begin{aligned} \Omega_{BA}^B &= \left(\mathbf{R}_{B(t)}^A \right)^T \dot{\mathbf{R}}_{B(t)}^A \\ &= (\mathbf{R}_1 \mathbf{R}_2 \mathbf{R}_3)^T \overbrace{\dot{\mathbf{R}}_1 \mathbf{R}_2 \mathbf{R}_3} \\ &= \mathbf{R}_3^T \mathbf{R}_2^T \mathbf{R}_1^T \left[\dot{\mathbf{R}}_1 \mathbf{R}_2 \mathbf{R}_3 + \mathbf{R}_1 \dot{\mathbf{R}}_2 \mathbf{R}_3 + \mathbf{R}_1 \mathbf{R}_2 \dot{\mathbf{R}}_3 \right] \\ &= \mathbf{R}_3^T \mathbf{R}_2^T \mathbf{R}_1^T \dot{\mathbf{R}}_1 \mathbf{R}_2 \mathbf{R}_3 + \mathbf{R}_3^T \mathbf{R}_2^T \dot{\mathbf{R}}_2 \mathbf{R}_3 + \mathbf{R}_3^T \dot{\mathbf{R}}_3 \\ &= \mathbf{R}_3^T \mathbf{R}_2^T [\mathbf{i}_1]_{\times} \dot{\varphi} \mathbf{R}_2 \mathbf{R}_3 + \mathbf{R}_3^T [\mathbf{i}_2]_{\times} \dot{\theta} \mathbf{R}_3 + [\mathbf{i}_3]_{\times} \dot{\psi} \\ &= [\mathbf{R}_3^T \mathbf{R}_2^T \mathbf{i}_1 \dot{\varphi}]_{\times} + [\mathbf{R}_3^T \mathbf{i}_2 \dot{\theta}]_{\times} + [\mathbf{i}_3 \dot{\psi}]_{\times}. \end{aligned} \quad (312)$$

Therefore, we have:

$$\omega_{BA}^B = \begin{bmatrix} \mathbf{R}_3^T \mathbf{R}_2^T \mathbf{i}_1 & \mathbf{R}_3^T \mathbf{i}_2 & \mathbf{i}_3 \end{bmatrix} \begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \mathbf{S}_{123}(\theta(t), \psi(t)) \dot{\Psi}_{BA}, \quad (313)$$

where we defined $\mathbf{S}_{123}(\theta, \psi) = [\mathbf{R}_3^T \mathbf{R}_2^T \mathbf{i}_1, \mathbf{R}_3^T \mathbf{i}_2, \mathbf{i}_3] \in \mathbb{R}^{3 \times 3}$. Equation (313) relates the angular velocity in terms of the Euler angles and the Euler rates. We have:

$$\mathbf{S}_{123}(\theta(t), \psi(t)) = \begin{bmatrix} \cos \theta(t) \cos \psi(t) & \sin \psi(t) & 0 \\ -\cos \theta(t) \sin \psi(t) & \cos \psi(t) & 0 \\ \sin \theta(t) & 0 & 1 \end{bmatrix}. \quad (314)$$

The inverse relationship can be computed by finding the inverse \mathbf{S}_{123}^{-1} :

$$\begin{aligned} \dot{\Psi}_{BA} &= \mathbf{S}_{123}^{-1}(\theta, \psi) \boldsymbol{\omega}_{BA}^B \\ &= \begin{bmatrix} \sec \theta(t) \cos \psi(t) & -\sec \theta(t) \sin \psi(t) & 0 \\ \sin \psi(t) & \cos \psi(t) & 0 \\ -\tan \theta(t) \cos \psi(t) & \tan \theta(t) \sin \psi(t) & 1 \end{bmatrix} \boldsymbol{\omega}_{BA}^B. \end{aligned} \quad (315)$$

Note that \mathbf{S}_{123}^{-1} does not exist at $\theta = \pi/2$.

V Differential Structure of $SO(3)$

The time derivative of rotation matrices derived in (295) using vectrix derivatives was kinematic in flavor. The same result can be obtained from an algebraic angle. This part introduces the Lie group structure of $SO(3)$, its associated Lie algebra $\mathfrak{so}(3)$, the exponential and logarithmic maps connecting the group and the algebra, and the adjoint representation that governs the frame changes in the Lie algebra. The treatment here is not meant to be rigorous or exhaustive and readers are strongly encouraged to refer to [Hal15, Tu11, SDA18] for a more thorough treatment.

10 Matrix Lie Group and Lie Algebra

10.1 Lie Groups

Elements of $SO(3)$ possess more than a discrete algebraic structure. Sections 2 and 3 showed that rotations satisfy the axioms of group under composition and vary continuously and smoothly. In addition, as discussed in Section 3.4, $SO(3)$ is a smooth manifold. A *Lie group* is a group that is also a smooth manifold. Therefore, $SO(3)$ is a Lie group.

10.1.1 $SO(3)$ as a Lie Group

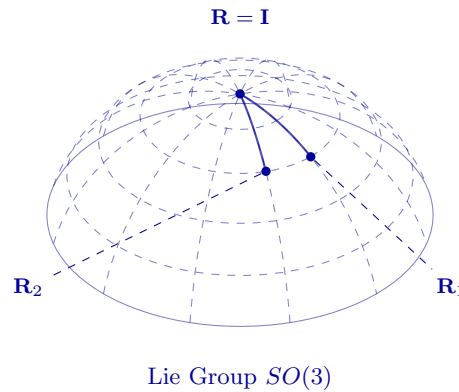


Figure 6: Visualization of $SO(3)$ as a smooth manifold

A *Lie group* is a group (G, \star) that is also a finite-dimensional smooth manifold such that the group operations – composition and inversion – are smooth maps:

$$\mu : G \times G \rightarrow G, \quad \mu(g_1, g_2) = g_1 \star g_2, \quad (316)$$

$$\iota : G \rightarrow G, \quad \iota(g) = g^{-1}. \quad (317)$$

The smoothness requirement ensures that combining or inverting group elements depends smoothly on the elements themselves. This is what distinguishes a Lie group from a normal topological group, i.e., not only are the group operations continuous, they are *infinitely differentiable*.

Lie groups arise naturally whenever a continuous family of symmetries or transformations is present. The group structure captures how transformations compose and invert, while the smooth manifold structure captures how transformations vary continuously and allows the use of calculus. $SO(3)$ is both a group and a smooth manifold, and the group operations are smooth. Therefore, $SO(3)$ is a Lie group.

Figure 6 illustrates $SO(3)$ as a smooth manifold and Table 7 summarizes its properties.

Property	$SO(3)$
Elements	Realized as 3×3 orthogonal matrices
Composition	Matrix multiplication: $\mathbf{R}_1 \mathbf{R}_2$
Identity	\mathbf{I}
Inverse	$\mathbf{R}^{-1} = \mathbf{R}^T$
Dimension	3
Compact	Yes (Section 3.3.1)
Connected	Yes (Section 3.3.2)
Simply connected	No ($\pi_1(SO(3)) \cong \mathbb{Z}_2$, Section 3.3.3)

Table 7: Summary of $SO(3)$ as a Lie group

10.2 Lie Algebra

Every Lie group has an associated Lie algebra that captures its infinitesimal structure. For $SO(3)$, this Lie algebra is the tangent space at the identity element and provides the vector space in which angular velocities, perturbations, and derivatives naturally live.

10.2.1 Tangent Space at the Identity

Let $\mathbf{R}(t) \in SO(3)$ be a smooth curve such that $\mathbf{R}(0) = \mathbf{I}$. Since $\mathbf{R}(t)$ remains in $SO(3)$ for all t , it satisfies the orthogonality condition:

$$\mathbf{R}(t)^T \mathbf{R}(t) = \mathbf{I}. \quad (318)$$

Differentiating with respect to t gives:

$$\dot{\mathbf{R}}(t)^T \mathbf{R}(t) + \mathbf{R}(t)^T \dot{\mathbf{R}}(t) = \mathbf{0}. \quad (319)$$

Evaluating at $t = 0$ and using $\mathbf{R}(0) = \mathbf{I}$ yields:

$$\dot{\mathbf{R}}(0)^T + \dot{\mathbf{R}}(0) = \mathbf{0}, \quad (320)$$

which shows that $\dot{\mathbf{R}}(0)$ is *skew-symmetric*: it equals the negative of its transpose. Since this holds for every smooth curve through the identity, the tangent space of $SO(3)$ at the identity element \mathbf{I} is the set of all 3×3 skew-symmetric matrices:

$$T_{\mathbf{I}}SO(3) = \mathfrak{so}(3) := \{ \boldsymbol{\Omega} \in \mathbb{R}^{3 \times 3} \mid \boldsymbol{\Omega}^T = -\boldsymbol{\Omega} \}. \quad (321)$$

This is a 3-dimensional vector space, consistent with $SO(3)$ being a 3-dimensional manifold. Every $\boldsymbol{\Omega} \in T_{\mathbf{I}}SO(3)$ can be written as:

$$\boldsymbol{\Omega} = [\boldsymbol{\omega}]_{\times} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}, \quad \boldsymbol{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \in \mathbb{R}^3. \quad (322)$$

10.2.2 Tangent Space at a General Point

Let $\mathbf{R}(t) \in SO(3)$ be a smooth curve. From (319):

$$\mathbf{R}(t)^T \dot{\mathbf{R}}(t) = -(\mathbf{R}(t)^T \dot{\mathbf{R}}(t))^T. \quad (323)$$

Hence, the matrix:

$$\boldsymbol{\Omega}(t) := \mathbf{R}(t)^T \dot{\mathbf{R}}(t), \quad (324)$$

is skew-symmetric and therefore belongs to $\mathfrak{so}(3)$ for all t . Left-multiplying by $\mathbf{R}(t)$ and using $\mathbf{R}(t)\mathbf{R}(t)^T = \mathbf{I}$ yields:

$$\dot{\mathbf{R}}(t) = \mathbf{R}(t)\boldsymbol{\Omega}(t), \quad \boldsymbol{\Omega}(t) \in \mathfrak{so}(3). \quad (325)$$

Therefore, the tangent space at a general point $\mathbf{R} \in SO(3)$ is obtained by left translation of the tangent space at the identity:

$$T_{\mathbf{R}}SO(3) = \{\mathbf{R}\boldsymbol{\Omega} \mid \boldsymbol{\Omega} \in T_{\mathbf{I}}SO(3)\}. \quad (326)$$

This trivialization of the *tangent bundle* (i.e., collection of all of the tangent spaces for all points on a manifold) by left translation is a defining feature of Lie groups. Comparing (325) with the kinematic result obtained in (295) via vectrix derivatives:

$$\dot{\mathbf{R}}_{B(t)}^A = \mathbf{R}_{B(t)}^A \boldsymbol{\Omega}_{BA}^B, \quad (327)$$

shows that the angular velocity skew matrix $\boldsymbol{\Omega}_{BA}^B$ is naturally an element of $\mathfrak{so}(3)$, while $\dot{\mathbf{R}}_{B(t)}^A$ lies in the left-translated tangent space $T_{\mathbf{R}_{B(t)}^A}SO(3)$. In this sense, the kinematic formulation from Section 9.1 is therefore the Lie-theoretic statement that angular velocity generates motion in the tangent space along the curve $\mathbf{R}_{B(t)}^A$.

10.2.3 The Lie Algebra $\mathfrak{so}(3)$

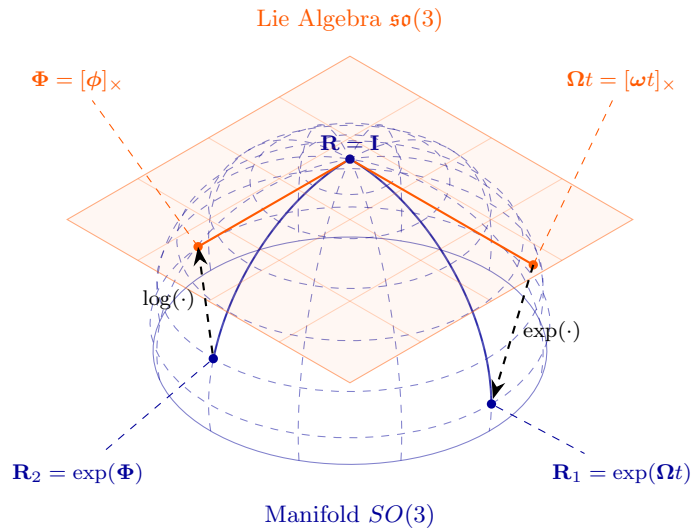


Figure 7: Visualization of the Lie group $SO(3)$ and its Lie algebra $\mathfrak{so}(3)$. The Lie algebra $\mathfrak{so}(3)$ is the tangent space at the identity $\mathbf{R} = \mathbf{I}$. The geodesic path on $SO(3)$ through the origin can be related to the straight path through the origin on the $\mathfrak{so}(3)$ through exponential and logarithmic mapping.

Formally, a *Lie algebra* is a vector space \mathbb{V} over a field \mathbb{F} , equipped with a binary operator $[\cdot, \cdot] : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{V}$ called the *Lie bracket*, satisfying the following axioms:

1. *Closure*: $\forall \mathbf{X}, \mathbf{Y} \in \mathbb{V}, [\mathbf{X}, \mathbf{Y}] \in \mathbb{V}$.
2. *Bilinearity*: $\forall \mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathbb{V}, \forall a, b \in \mathbb{F}$:

$$[a\mathbf{X} + b\mathbf{Y}, \mathbf{Z}] = a[\mathbf{X}, \mathbf{Z}] + b[\mathbf{Y}, \mathbf{Z}], \quad (328)$$

$$[\mathbf{Z}, a\mathbf{X} + b\mathbf{Y}] = a[\mathbf{Z}, \mathbf{X}] + b[\mathbf{Z}, \mathbf{Y}]. \quad (329)$$

3. *Alternating*: $\forall \mathbf{X} \in \mathbb{V}, [\mathbf{X}, \mathbf{X}] = \mathbf{0}$.
4. *Jacobi identity*: $\forall \mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathbb{V}$:

$$[\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]] + [\mathbf{Z}, [\mathbf{X}, \mathbf{Y}]] + [\mathbf{Y}, [\mathbf{Z}, \mathbf{X}]] = \mathbf{0}. \quad (330)$$

The alternating property together with bilinearity implies *anti-commutativity*: $[\mathbf{X}, \mathbf{Y}] = -[\mathbf{Y}, \mathbf{X}]$. Note that a Lie algebra is, in particular, a vector space, so addition, scalar multiplication, and the zero element are all well-defined.

Every Lie group G has an associated Lie algebra \mathfrak{g} , defined as the tangent space of G at the identity element, equipped with a Lie bracket derived from the group structure. The Lie algebra captures the local, infinitesimal behavior of the group, i.e., while the group describes finite transformations, its Lie algebra describes infinitesimal ones.

The Lie algebra of $SO(3)$, denoted $\mathfrak{so}(3)$, is the tangent space at the identity characterized in (321):

$$\mathfrak{so}(3) := T_{\mathbf{I}}SO(3) = \left\{ \boldsymbol{\Omega} \in \mathbb{R}^{3 \times 3} \mid \boldsymbol{\Omega}^T = -\boldsymbol{\Omega} \right\}, \quad (331)$$

regarded as a 3-dimensional vector space over \mathbb{R} and equipped with the matrix commutator as its Lie bracket:

$$[\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2] := \boldsymbol{\Omega}_1 \boldsymbol{\Omega}_2 - \boldsymbol{\Omega}_2 \boldsymbol{\Omega}_1, \quad \boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2 \in \mathfrak{so}(3). \quad (332)$$

This bracket satisfies the Lie algebra axioms:

1. *Closure*: The commutator of two skew-symmetric matrices is skew-symmetric. Let $\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2 \in \mathfrak{so}(3)$:

$$[\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2]^T = (\boldsymbol{\Omega}_1 \boldsymbol{\Omega}_2 - \boldsymbol{\Omega}_2 \boldsymbol{\Omega}_1)^T = \boldsymbol{\Omega}_2^T \boldsymbol{\Omega}_1^T - \boldsymbol{\Omega}_1^T \boldsymbol{\Omega}_2^T = \boldsymbol{\Omega}_2 \boldsymbol{\Omega}_1 - \boldsymbol{\Omega}_1 \boldsymbol{\Omega}_2 = -[\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2]. \quad (333)$$

2. *Bilinearity*: Follows from the linearity of matrix multiplication.
3. *Alternating*: $[\boldsymbol{\Omega}, \boldsymbol{\Omega}] = \boldsymbol{\Omega}^2 - \boldsymbol{\Omega}^2 = \mathbf{0}$.
4. *Jacobi identity*: Follows by direct expansion of the matrix commutators.

Under the correspondence $\boldsymbol{\Omega} = [\boldsymbol{\omega}]_{\times}$ established in (322), the Lie bracket on $\mathfrak{so}(3)$ maps to the cross product on \mathbb{R}^3 :

$$[[\boldsymbol{\omega}_1]_{\times}, [\boldsymbol{\omega}_2]_{\times}] = [\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2]_{\times}. \quad (334)$$

This follows from direct expansion of the matrix commutator. Therefore, (\mathbb{R}^3, \times) is a Lie algebra isomorphic to $\mathfrak{so}(3)$. The cross product on \mathbb{R}^3 is the Lie bracket of the rotation group. This isomorphism is what allows every subsequent Lie-theoretic computation on $SO(3)$ to be carried out in terms of three-dimensional vectors and cross products rather than 3×3 matrix commutators.

10.3 Elements of $\mathfrak{so}(3)$

10.3.1 Generators and Displacements

The Lie algebra $\mathfrak{so}(3)$ defined in (331) is a vector space: it consists of all 3×3 skew-symmetric matrices. Any vector $\boldsymbol{\tau} = [\tau_1, \tau_2, \tau_3] \in \mathbb{R}^3$ produces an element $[\boldsymbol{\tau}]_{\times} \in \mathfrak{so}(3)$, regardless of its physical meaning.

We can write:

$$[\boldsymbol{\tau}]_{\times} = \begin{bmatrix} 0 & -\tau_3 & \tau_2 \\ \tau_3 & 0 & -\tau_1 \\ -\tau_2 & \tau_1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}}_{\mathbf{G}_1} \tau_1 + \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}}_{\mathbf{G}_2} \tau_2 + \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{G}_3} \tau_3, \quad (335)$$

where $\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3$ are called the *generators* of $\mathfrak{so}(3)$. Therefore, the elements of $\mathfrak{so}(3)$ can be written as the linear combination of the generators specified by a vector of coefficients $\boldsymbol{\tau}$. The canonical elements were identified as the skew-symmetric matrix form of angular velocity vectors $\boldsymbol{\omega}$ but rotations vectors $\boldsymbol{\phi}$ are also valid:

$$\boldsymbol{\Omega} := [\boldsymbol{\omega}]_{\times} \in \mathfrak{so}(3), \quad \boldsymbol{\Phi} := [\boldsymbol{\phi}]_{\times} \in \mathfrak{so}(3). \quad (336)$$

Algebraically, both are skew-symmetric matrices and both belong to $\mathfrak{so}(3)$. The difference is in their physical roles:

1. The angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ is a *velocity*: it is the tangent vector to a curve in $SO(3)$ and describes the instantaneous rate of rotation. It has units of radians per second and was derived in Section 10.2.2 as the quantity satisfying $\dot{\mathbf{R}} = \mathbf{R}[\boldsymbol{\omega}]_{\times}$.
2. The rotation vector $\boldsymbol{\phi} = \mathbf{u}\theta \in \mathbb{R}^3$ is a *displacement*: it parameterizes a finite rotation by angle θ about axis \mathbf{u} , as introduced in Section 5. It has units of radians and corresponds to a definite element of $SO(3)$ via the exponential map.

Therefore, the exponential map that connects $\mathfrak{so}(3)$ to $SO(3)$ takes as its argument not $\boldsymbol{\Omega}$ but the skew matrix form of the rotation vector $\boldsymbol{\Phi}$. The connection between them was established in Section 8.2 as $\boldsymbol{\phi}(t) = \boldsymbol{\omega} t$ for $\boldsymbol{\omega}$ over an interval $[0, t]$. This constant angular velocity case motivates the exponential map. Given the constant angular velocity $[\boldsymbol{\omega}]_{\times} \in \mathfrak{so}(3)$ and a duration t , the rotation $\mathbf{R} \in SO(3)$ resulting from applying the angular velocity for time t can be obtained using *one-parameter subgroup*.

10.3.2 One-Parameter Subgroup

Recall from Section 9.1 that the orientation of frame F_B relative to frame F_A satisfies the kinematic ODE:

$$\dot{\mathbf{R}}_{B(t)}^A = \mathbf{R}_{B(t)}^A \boldsymbol{\Omega}_{BA}^B, \quad \boldsymbol{\Omega}_{BA}^B = [\boldsymbol{\omega}_{BA}^B]_{\times} \in \mathfrak{so}(3). \quad (337)$$

The angular velocity is assumed to be constant, i.e., $\boldsymbol{\omega}_{BA}^B(t) \equiv \boldsymbol{\omega}_{BA}^B$, and write $\boldsymbol{\Omega} := [\boldsymbol{\omega}_{BA}^B]_{\times}$ for brevity. To solve (337), define:

$$\mathbf{Q}(t) := \mathbf{R}_{B(t)}^A \exp(-\boldsymbol{\Omega} t). \quad (338)$$

Differentiating with respect to t :

$$\begin{aligned} \dot{\mathbf{Q}}(t) &= \dot{\mathbf{R}}_{B(t)}^A \exp(-\boldsymbol{\Omega} t) + \mathbf{R}_{B(t)}^A \frac{d}{dt} \exp(-\boldsymbol{\Omega} t) \\ &= \dot{\mathbf{R}}_{B(t)}^A \exp(-\boldsymbol{\Omega} t) - \mathbf{R}_{B(t)}^A \boldsymbol{\Omega} \exp(-\boldsymbol{\Omega} t) \quad \left(\frac{d}{dt} \exp(\mathbf{A} t) = \mathbf{A} \exp(\mathbf{A} t) \right) \\ &= \mathbf{R}_{B(t)}^A \boldsymbol{\Omega} \exp(-\boldsymbol{\Omega} t) - \mathbf{R}_{B(t)}^A \boldsymbol{\Omega} \exp(-\boldsymbol{\Omega} t) \quad \left(\dot{\mathbf{R}}_{B(t)}^A = \mathbf{R}_{B(t)}^A \boldsymbol{\Omega} \right) \\ &= \mathbf{0}, \end{aligned} \quad (339)$$

so $\mathbf{Q}(t)$ is constant: $\mathbf{Q}(t) = \mathbf{Q}(0) = \mathbf{R}_{B(0)}^A$. Unfolding the definition yields the closed-form solution:

$$\boxed{\mathbf{R}_{B(t)}^A = \mathbf{R}_{B(0)}^A \exp(\boldsymbol{\Omega}_{BA}^B t) = \mathbf{R}_{B(0)}^A \exp\left([\boldsymbol{\omega}_{BA}^B t]_{\times}\right)}. \quad (340)$$

This establishes the result stated without proof in Section 9.1. Most importantly, the argument of the exponential is $[\boldsymbol{\omega}t]_{\times} = [\boldsymbol{\phi}(t)]_{\times}$. Therefore, the exponential map converts finite rotation vector (Lie algebra element) into a rotation matrix (Lie group element). If F_A and F_B are initially aligned, $\mathbf{R}_{B(0)}^A = \mathbf{I}$, and the trajectory reduces to:

$$\mathbf{R}_{B(t)}^A = \exp([\boldsymbol{\omega}_{BA}^B]_{\times} t). \quad (341)$$

The map $\gamma : \mathbb{R} \rightarrow SO(3)$ defined by $\gamma(t) := \exp(\boldsymbol{\Omega}t)$ is called the *one-parameter subgroup generated by $\boldsymbol{\Omega}$* . It satisfies the *group homomorphism* property:

$$\gamma(s+t) = \exp(\boldsymbol{\Omega}(s+t)) = \exp(\boldsymbol{\Omega}s)\exp(\boldsymbol{\Omega}t) = \gamma(s)\gamma(t), \quad s, t \in \mathbb{R}, \quad (342)$$

which holds because $\boldsymbol{\Omega}s$ and $\boldsymbol{\Omega}t$ commute (both are scalar multiples of the same skew-symmetric matrix). Every element $\boldsymbol{\Omega} \in \mathfrak{so}(3)$ generates such a subgroup, and conversely, every smooth homomorphism $\mathbb{R} \rightarrow SO(3)$ arises in this way.

Geometrically, a one-parameter subgroup is a *geodesic* through the identity on $SO(3)$ with respect to the bi-invariant metric. It gives the locally shortest path from \mathbf{I} to nearby points $\exp(\boldsymbol{\Omega}t)$. The general solution (340) is obtained by left-translation of this geodesic so that it begins at $\mathbf{R}_{B(0)}^A$. Constant angular velocity motion on $SO(3)$ is therefore constant-speed motion along a geodesic – the rotational analogue of straight-line motion at constant velocity in Euclidean space.

10.4 Exponential and Logarithmic Maps

The Lie algebra $\mathfrak{so}(3)$ captures the infinitesimal structure of $SO(3)$, but practical applications require moving between the Lie algebra and the Lie group—between infinitesimal and finite rotations. The *exponential map* and its inverse, the *logarithmic map*, provide this bridge.

10.4.1 Exponential Map: $\mathfrak{so}(3) \rightarrow SO(3)$

As discussed in Section 5.1.1, the *exponential map* for $SO(3)$ maps a Lie algebra element $[\boldsymbol{\phi}]_{\times} \in \mathfrak{so}(3)$ to a finite rotation $\mathbf{R} \in SO(3)$, where $\boldsymbol{\phi} = \mathbf{u}\theta \in \mathbb{R}^3$ is the rotation vector. Formally, the exponential map for $SO(3)$ is defined as:

$$\exp : \mathfrak{so}(3) \rightarrow SO(3), \quad [\boldsymbol{\phi}]_{\times} \mapsto \exp([\boldsymbol{\phi}]_{\times}). \quad (343)$$

Following [Sol17], we also write $\text{Exp} : \mathbb{R}^3 \rightarrow SO(3)$ for the composition of the skew operator and the matrix exponential:

$$\text{Exp}(\boldsymbol{\phi}) := \exp([\boldsymbol{\phi}]_{\times}). \quad (344)$$

To verify that $\text{Exp}(\boldsymbol{\phi}) \in SO(3)$, note that $[\boldsymbol{\phi}]_{\times}$ is skew-symmetric and therefore:

$$\exp([\boldsymbol{\phi}]_{\times})^T = \exp([\boldsymbol{\phi}]_{\times}^T) = \exp(-[\boldsymbol{\phi}]_{\times}) = \exp([\boldsymbol{\phi}]_{\times})^{-1}, \quad (345)$$

which shows that $\exp([\boldsymbol{\phi}]_{\times})$ is orthogonal. Since $\det(\exp([\boldsymbol{\phi}]_{\times})) = \exp(\text{tr}([\boldsymbol{\phi}]_{\times}))$ and $\text{tr}([\boldsymbol{\phi}]_{\times}) = 0$, the determinant is 1. Hence $\text{Exp}(\boldsymbol{\phi}) \in SO(3)$.

The closed-form expression for the exponential map is derived in Section 5.1.1 with *Rodrigues'* formula by exploiting the algebraic properties of skew-symmetric matrices:

$$\boxed{\text{Exp}(\boldsymbol{\phi}) = \exp([\boldsymbol{\phi}]_{\times}) = \mathbf{I} + \sin \theta [\mathbf{u}]_{\times} + (1 - \cos \theta) [\mathbf{u}]_{\times}^2}, \quad (346)$$

where $\theta = \|\boldsymbol{\phi}\|$ and $\mathbf{u} = \boldsymbol{\phi}/\theta$. The exponential map $\exp : \mathfrak{so}(3) \rightarrow SO(3)$ is *surjective*: every rotation matrix $\mathbf{R} \in SO(3)$ can be written as $\text{Exp}(\boldsymbol{\phi})$ for some $\boldsymbol{\phi} \in \mathbb{R}^3$. The exponential map thus provides the Lie-theoretic interpretation of *Rodrigues'* formula.

In a neighborhood of $\boldsymbol{\phi} = \mathbf{0}$ (i.e., near the identity rotation), the exponential map is a *local diffeomorphism*: it is smooth, bijective, and has a smooth inverse (i.e., the logarithmic map). This means that near the identity, rotations can be parameterized by vectors in \mathbb{R}^3 without ambiguity or singularity.

More precisely, the exponential map restricted to the open ball $\{\boldsymbol{\phi} \in \mathbb{R}^3 \mid \|\boldsymbol{\phi}\| < \pi\}$ is a diffeomorphism onto its image. This provides a smooth, bijective local chart for $SO(3)$ near any rotation \mathbf{R}_0 , by composing with left or right multiplication:

$$\boldsymbol{\phi} \mapsto \mathbf{R}_0 \exp([\boldsymbol{\phi}]_{\times}), \quad \|\boldsymbol{\phi}\| < \pi. \quad (347)$$

This local parameterization is the mathematical basis for perturbation-based approaches in state estimation, where a rotation is represented as a reference rotation composed with a small correction in the tangent space.

The exponential map is *not injective* globally: distinct elements of $\mathfrak{so}(3)$ can map to the same rotation. There are two sources of non-injectivity:

1. *Periodicity*: For any unit axis \mathbf{u} , rotations by θ and $\theta + 2k\pi$ ($k \in \mathbb{Z}$) are identical:

$$\exp([\theta\mathbf{u}]_{\times}) = \exp([\theta + 2k\pi]\mathbf{u}]_{\times}), \quad k \in \mathbb{Z}. \quad (348)$$

2. *Antipodal ambiguity at $\theta = \pi$* : As discussed in Section 3.2.1, $\exp([\pi\mathbf{u}]_{\times}) = \exp([\pi\mathbf{u}]_{\times})$. This is the same antipodal identification responsible for the non-simple-connectedness of $SO(3)$.

For small angles $\delta\theta$, the first-order approximation can be computed as:

$$\exp([\boldsymbol{\phi}]_{\times}) \approx \mathbf{I} + [\delta\boldsymbol{\phi}]_{\times}. \quad (349)$$

10.4.2 Logarithmic Map: $SO(3) \rightarrow \mathfrak{so}(3)$

We have defined the logarithmic map in Section 5.2.1. The *logarithmic map* is the inverse of the exponential map. Given a rotation matrix $\mathbf{R} \in SO(3)$, it returns the corresponding element of $\mathfrak{so}(3)$:

$$\log : SO(3) \rightarrow \mathfrak{so}(3), \quad \mathbf{R} \mapsto [\boldsymbol{\phi}]_{\times}, \quad (350)$$

where $\boldsymbol{\phi} = \theta\mathbf{u}$ is the rotation vector. Similarly, we write $\text{Log} : SO(3) \rightarrow \mathbb{R}^3$ for the composition with the reverse skew operator:

$$\text{Log}(\mathbf{R}) := (\log(\mathbf{R}))^{\vee} = \boldsymbol{\phi} \in \mathbb{R}^3. \quad (351)$$

The rotation angle θ and the unit axis of rotation \mathbf{u} are recovered from \mathbf{R} as derived in Section 5.2.1:

$$\text{tr}(\mathbf{R}) = 1 + 2 \cos \theta, \quad \theta = \arccos\left(\frac{\text{tr}(\mathbf{R}) - 1}{2}\right), \quad \theta \in [0, \pi]. \quad (352)$$

For $\theta \neq 0$ and $\theta \neq \pi$, the skew-symmetric part of \mathbf{R} yields:

$$[\mathbf{u}]_{\times} = \frac{\mathbf{R} - \mathbf{R}^T}{2 \sin \theta}, \quad \mathbf{u} = \left(\frac{\mathbf{R} - \mathbf{R}^T}{2 \sin \theta}\right)^{\vee}. \quad (353)$$

Note that if $\theta = 0$, then $\mathbf{R} = \mathbf{I}$ and $\boldsymbol{\phi} = \mathbf{0}$. Furthermore, (353) is undefined for $\theta = \pi$. The axis \mathbf{u} must be found as the unit eigenvector of \mathbf{R} corresponding to eigenvalue +1. Moreover, \mathbf{u} and $-\mathbf{u}$ are both valid—the logarithmic map is not unique at $\theta = \pi$.

10.5 Adjoints

Vectors of the tangent space at one element can be transformed to the tangent space at of another through a linear transformation. This transform is called the *adjoint*. The adjoint representations are summarized in Table 8.

	Definition	Matrix form on \mathbb{R}^3
$\text{Ad}_{\mathbf{R}}$	$\mathbf{R}[\boldsymbol{\omega}]_{\times} \mathbf{R}^T = [\mathbf{R}\boldsymbol{\omega}]_{\times}$	$\mathbf{Ad}_{\mathbf{R}} = \mathbf{R}$
$\text{ad}_{\boldsymbol{\omega}_1}$	$[[\boldsymbol{\omega}_1]_{\times}, [\boldsymbol{\omega}_2]_{\times}] = [\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2]_{\times}$	$\mathbf{ad}_{\boldsymbol{\omega}} = [\boldsymbol{\omega}]_{\times}$

Table 8: Adjoint representations of $SO(3)$ and $\mathfrak{so}(3)$.

10.5.1 Adjoint of $SO(3)$

For each $\mathbf{R} \in SO(3)$, the *adjoint map* $\text{Ad}_{\mathbf{R}} : \mathfrak{so}(3) \rightarrow \mathfrak{so}(3)$ is defined by conjugation:

$$\text{Ad}_{\mathbf{R}}([\boldsymbol{\omega}]_{\times}) := \mathbf{R}[\boldsymbol{\omega}]_{\times} \mathbf{R}^T = [\mathbf{R}\boldsymbol{\omega}]_{\times}, \quad \boldsymbol{\omega} \in \mathbb{R}^3. \quad (354)$$

Hence, the adjoint action on \mathbb{R}^3 is simply multiplication by \mathbf{R} :

$$\text{Ad}_{\mathbf{R}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad \boldsymbol{\omega} \mapsto \mathbf{R}\boldsymbol{\omega}. \quad (355)$$

The *adjoint matrix* of $SO(3)$ is therefore \mathbf{R} itself:

$$\mathbf{Ad}_{\mathbf{R}} = \mathbf{R} \in \mathbb{R}^{3 \times 3}. \quad (356)$$

Physically, this is a change of resolving frame for an angular velocity. The adjoint representation is the Lie-theoretic formalization of this frame change.

10.5.2 Adjoint of $\mathfrak{so}(3)$

The Lie algebra also has an adjoint action, defined via the Lie bracket. For each $[\boldsymbol{\omega}_1]_{\times} \in \mathfrak{so}(3)$, the *adjoint map* $\text{ad}_{[\boldsymbol{\omega}_1]_{\times}} : \mathfrak{so}(3) \rightarrow \mathfrak{so}(3)$ is:

$$\text{ad}_{[\boldsymbol{\omega}_1]_{\times}}([\boldsymbol{\omega}_2]_{\times}) := [[\boldsymbol{\omega}_1]_{\times}, [\boldsymbol{\omega}_2]_{\times}] = [\boldsymbol{\omega}_1]_{\times}[\boldsymbol{\omega}_2]_{\times} - [\boldsymbol{\omega}_2]_{\times}[\boldsymbol{\omega}_1]_{\times} = [\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2]_{\times}. \quad (357)$$

On \mathbb{R}^3 , the adjoint action is:

$$\text{ad}_{\boldsymbol{\omega}_1}(\boldsymbol{\omega}_2) = \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2 = [\boldsymbol{\omega}_1]_{\times} \boldsymbol{\omega}_2. \quad (358)$$

Therefore, the *adjoint matrix* of $\mathfrak{so}(3)$ is:

$$\mathbf{ad}_{\boldsymbol{\omega}} = [\boldsymbol{\omega}]_{\times} \in \mathbb{R}^{3 \times 3}. \quad (359)$$

10.5.3 Visualizing the Adjoins

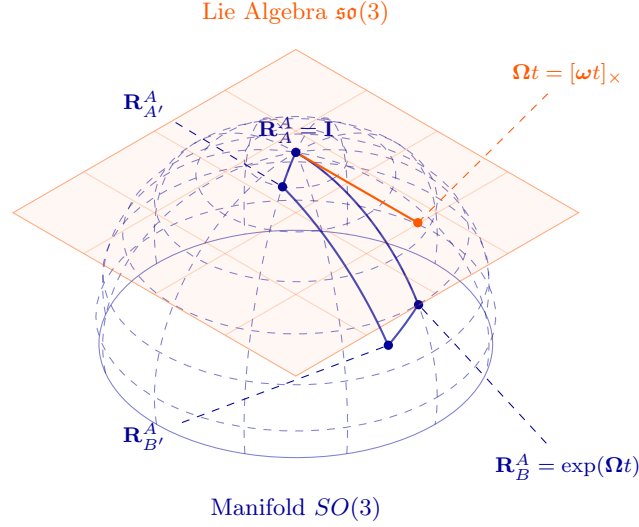


Figure 8: Visualization of the Lie group $SO(3)$ and its Lie algebra $\mathfrak{so}(3)$. The Lie algebra $\mathfrak{so}(3)$ is the tangent space at the identity $\mathbf{R} = \mathbf{I}$. The geodesic path on $SO(3)$ through the origin can be related to the straight path through the origin on the $\mathfrak{so}(3)$ through exponential and logarithmic mapping.

Let F_A be a frame. The tangent space at the identity $\mathbf{R}_A^A = \mathbf{I}$ is illustrated in Figure 8. Any rotation relative to F_A can be drawn as a point on the smooth manifold. For example, given frame F_B , the rotation \mathbf{R}_B^A is computed using $\Omega^A \in \mathfrak{so}(3)$ as:

$$\mathbf{R}_B^A = \text{Exp}(\Omega^A t). \quad (360)$$

If \mathbf{R}_B^A is perturbed *locally*, i.e., on the tangent space at \mathbf{R}_B^A by $\delta\Omega^B$ over δt , then we have:

$$\mathbf{R}_{B'}^A = \mathbf{R}_B^A \mathbf{R}_{B'}^B = \text{Exp}(\Omega^A t) \text{Exp}(\delta\Omega^B \delta t). \quad (361)$$

Alternatively, the same perturbed point can be described by holding F_B fixed and perturbing the reference frame instead. That is, we define a perturbation $\delta\Omega^A$ on the tangent space at the identity that takes F_A to a new frame $F_{A'}$ over the same duration δt :

$$\mathbf{R}_{A'}^A = \text{Exp}(\delta\Omega^A \delta t). \quad (362)$$

The point $\mathbf{R}_{B'}^A$, when re-expressed relative to the perturbed reference frame $F_{A'}$, becomes:

$$\mathbf{R}_B^{A'} = \mathbf{R}_{A'}^A \mathbf{R}_B^A = (\mathbf{R}_{A'}^A)^T \mathbf{R}_B^A = \text{Exp}(-\delta\Omega^A \delta t) \text{Exp}(\Omega^A t). \quad (363)$$

For both perturbations to describe the same point, we require $\mathbf{R}_B^{A'} = \mathbf{R}_{B'}^A$, which gives:

$$\text{Exp}(-\delta\Omega^A \delta t) \text{Exp}(\Omega^A t) = \text{Exp}(\Omega^A t) \text{Exp}(\delta\Omega^B \delta t). \quad (364)$$

Rearranging above yields:

$$\text{Exp}(-\delta\Omega^A \delta t) = \mathbf{R}_B^A \text{Exp}(\delta\Omega^B \delta t) (\mathbf{R}_B^A)^T. \quad (365)$$

Applying the conjugation identity $\mathbf{R} \text{Exp}(\phi) \mathbf{R}^T = \text{Exp}(\mathbf{R}\phi)$ for any $\mathbf{R} \in SO(3)$ and $\phi \in \mathfrak{so}(3)$:

$$\text{Exp}(-\delta\Omega^A \delta t) = \text{Exp}(\mathbf{R}_B^A \delta\Omega^B \delta t), \quad (366)$$

which yields the adjoint relationship between the two tangent vectors:

$$\delta\Omega^A = -\mathbf{R}_B^A \delta\Omega^B = -\mathbf{Ad}_{\mathbf{R}_B^A} \delta\Omega^B. \quad (367)$$

The negative sign reflects the duality of the two viewpoints. Perturbing the point \mathbf{R}_B^A forward by $\delta\Omega^B$ on the local tangent space is equivalent to perturbing the reference frame F_A backward by some $\delta\Omega^A$ on the tangent space at the identity. The two tangent vectors $\delta\Omega^A$ and $\delta\Omega^B$ describe the same physical infinitesimal rotation resolved in different frames, related by the adjoint $\mathbf{Ad}_{\mathbf{R}_B^A}$.

10.6 Composition, Difference, and Distance

Since $SO(3)$ is not a vector space, the concepts of composition, difference, and distance need to be defined differently using the exponential and logarithmic maps defined in Section 10.4.

10.6.1 Composition

Let $\mathbf{R} \in SO(3)$ and let $\phi \in \mathbb{R}^3$ be a tangent vector representing an incremental rotation. The *composition* of \mathbf{R} with ϕ , applied on the right, is the rotation:

$$\mathbf{R} \text{Exp}(\phi) \in SO(3). \quad (368)$$

Because $\text{Exp}(\phi) \in SO(3)$ by construction (Section 10.4.1) and $SO(3)$ is closed under multiplication, the result is always a valid rotation. This is the $SO(3)$ analogue of vector-space addition. It perturbs a rotation by a tangent vector while keeping the result on the manifold.

Physically, the rotation increment ϕ is resolved in the source frame of $\mathbf{R}_{\text{src}}^{\text{dst}}$. To see this, let F_A , F_B , and F_C be frames and let $\mathbf{R}_B^A, \mathbf{R}_C^A \in SO(3)$. Let $\phi_{CB}^C \in \mathbb{R}^3$ be a rotation vector of F_C relative to F_B . A composition on frame F_B composes \mathbf{R}_B^A as follows:

$$\boxed{\mathbf{R}_C^A = \mathbf{R}_B^A \text{Exp}(\phi_{CB}^C)}. \quad (369)$$

Note that $\phi_{CB}^C \equiv \phi_{CB}^B$: the rotation vector of \mathbf{R}_C^B is the common eigenvector of both \mathbf{R}_C^B and \mathbf{R}_B^C with eigenvalue +1, and therefore has the same components whether resolved in F_B or F_C (Section 5.3). By the exponential map, $\text{Exp}(\phi_{CB}^C) = \mathbf{R}_C^B$, so the composition reads $\mathbf{R}_C^A = \mathbf{R}_B^A \mathbf{R}_C^B$. The rotation-vector representation ϕ_{CB}^B is unique for $\|\phi_{CB}^B\| < \pi$, where the exponential map is a diffeomorphism (Section 10.4.1).

The alternative left composition,

$$\text{Exp}(\phi) \mathbf{R} \in SO(3), \quad (370)$$

interprets ϕ as a rotation vector resolved in the destination frame of $\mathbf{R}_{\text{src}}^{\text{dst}}$. To see this, let $\phi_{AC}^A \in \mathbb{R}^3$ be a rotation vector of F_A relative to F_C resolved in F_A . A composition on frame F_A composes \mathbf{R}_B^A as follows:

$$\boxed{\mathbf{R}_B^C = \text{Exp}(\phi_{AC}^A) \mathbf{R}_B^A}. \quad (371)$$

By the exponential map, $\text{Exp}(\phi_{AC}^A) = \mathbf{R}_A^C$, so the composition reads $\mathbf{R}_B^C = \mathbf{R}_A^C \mathbf{R}_B^A$.

10.6.2 Difference

Given two rotations $\mathbf{R}_1, \mathbf{R}_2 \in SO(3)$, the *rotation difference* (or *relative rotation vector*) is the tangent vector $\phi \in \mathbb{R}^3$ such that:

$$\mathbf{R}_2 = \mathbf{R}_1 \text{Exp}(\phi). \quad (372)$$

Solving for ϕ by left-multiplying with \mathbf{R}_1^T and applying the logarithmic map yields the explicit formula:

$$\phi = \text{Log}(\mathbf{R}_1^T \mathbf{R}_2) \in \mathbb{R}^3. \quad (373)$$

This is the $SO(3)$ analogue of vector-space subtraction. It returns the tangent-space increment that takes \mathbf{R}_1 to \mathbf{R}_2 . Because ϕ is obtained by composing $\mathbf{R}_1^T \mathbf{R}_2$, it is resolved in the source frame of \mathbf{R}_1 .

To see this, let F_A , F_B , and F_C be frames. Let $\mathbf{R}_C^A, \mathbf{R}_B^A \in SO(3)$ and let $\phi_{CB}^C \in \mathbb{R}^3$ such that $\mathbf{R}_C^A = \mathbf{R}_B^A \text{Exp}(\phi_{CB}^C)$. We have:

$$\phi_{CB}^C = \text{Log}\left(\left(\mathbf{R}_B^A\right)^T \mathbf{R}_C^A\right) = \text{Log}\left(\mathbf{R}_C^B\right). \quad (374)$$

The difference (373) inherits the non-injectivity of the logarithmic map (Section 10.4.2). It is well-defined when $\mathbf{R}_1^T \mathbf{R}_2$ has rotation angle $\theta < \pi$, and becomes ambiguous at $\theta = \pi$ due to the antipodal identification $\text{Exp}(\pi \mathbf{u}) = \text{Exp}(-\pi \mathbf{u})$. For rotations close to one another ($\theta \ll \pi$), the difference is unique and smoothly varying.

10.6.3 Distance

The *geodesic distance* between two rotations $\mathbf{R}_1, \mathbf{R}_2 \in SO(3)$ is defined as the magnitude of their rotation difference:

$$d(\mathbf{R}_1, \mathbf{R}_2) := \|\text{Log}(\mathbf{R}_1^T \mathbf{R}_2)\| = \|\phi\|, \quad (375)$$

where $\|\phi\| \in [0, \pi]$ is the rotation angle of the relative rotation $\mathbf{R}_1^T \mathbf{R}_2$. A closed form solution for the distance can be computed using (352):

$$d(\mathbf{R}_1, \mathbf{R}_2) = \arccos\left(\frac{\text{tr}(\mathbf{R}_1^T \mathbf{R}_2) - 1}{2}\right). \quad (376)$$

Geometrically, $d(\mathbf{R}_1, \mathbf{R}_2)$ is the length of the shortest curve in $SO(3)$ connecting \mathbf{R}_1 to \mathbf{R}_2 when $SO(3)$ is equipped with the bi-invariant Riemannian metric induced by the Frobenius inner product on $\mathfrak{so}(3)$:

$$\langle \boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2 \rangle := \frac{1}{2} \text{tr}(\boldsymbol{\Omega}_1^T \boldsymbol{\Omega}_2) = \boldsymbol{\omega}_1^T \boldsymbol{\omega}_2, \quad \boldsymbol{\Omega}_i = [\boldsymbol{\omega}_i]_{\times} \in \mathfrak{so}(3). \quad (377)$$

Under this metric, the shortest path from \mathbf{R}_1 to \mathbf{R}_2 is the one-parameter subgroup:

$$\gamma(s) = \mathbf{R}_1 \text{Exp}(s \text{Log}(\mathbf{R}_1^T \mathbf{R}_2)), \quad s \in [0, 1], \quad (378)$$

whose total length is exactly $\|\phi\|$. The distance (375) satisfies the standard metric axioms:

1. *Non-negativity*: $d(\mathbf{R}_1, \mathbf{R}_2) \geq 0$, with equality iff $\mathbf{R}_1 = \mathbf{R}_2$.
2. *Symmetry*: $d(\mathbf{R}_1, \mathbf{R}_2) = d(\mathbf{R}_2, \mathbf{R}_1)$, since $\mathbf{R}_2^T \mathbf{R}_1 = (\mathbf{R}_1^T \mathbf{R}_2)^T$ has the same rotation angle as $\mathbf{R}_1^T \mathbf{R}_2$.
3. *Triangle inequality*: $d(\mathbf{R}_1, \mathbf{R}_3) \leq d(\mathbf{R}_1, \mathbf{R}_2) + d(\mathbf{R}_2, \mathbf{R}_3)$.
4. *Bi-invariance*: $d(\mathbf{R}\mathbf{R}_1, \mathbf{R}\mathbf{R}_2) = d(\mathbf{R}_1, \mathbf{R}_2) = d(\mathbf{R}_1 \mathbf{R}, \mathbf{R}_2 \mathbf{R})$ for all $\mathbf{R} \in SO(3)$, since left or right multiplication preserves the trace of the relative rotation.

VI Differentiations and Integrations on $SO(3)$

Since $SO(3)$ is a Lie group and not a vector space, classical definitions of derivatives and integrals cannot be applied directly. However, the Lie algebra $\mathfrak{so}(3)$ is a vector space, and through the exponential and logarithmic maps along with the definitions of composition, difference, and distance developed in the previous part, analogous differential and algebraic operations can be defined on $SO(3)$. These tools are fundamental in state estimation, where the attitude state evolves on the space of orientations rather than in Euclidean space.

11 Perturbations on $SO(3)$

Let F_A and F_B be frames and let $\mathbf{R}_B^A \in SO(3)$. A *perturbed* rotation is a composition of an unperturbed rotation with a small perturbation. Let $F_{B'}$ be a frame obtained by perturbing F_B by a small rotation vector $\delta\phi_{B'B}^{B'} \in \mathbb{R}^3$. Using the definition of composition from Section 10.6.1, the *right perturbation* is defined as:

$$\mathbf{R}_{B'}^A = \mathbf{R}_B^A \text{Exp}(\delta\phi_{B'B}^{B'}). \quad (379)$$

The right perturbation perturbs the source frame of $\mathbf{R}_{\text{src}}^{\text{dst}}$ and the *left perturbation* perturbs the destination frame.

In state estimation applications, we often have the notion of local and global frames. A *local frame* (denoted by F_L) is a frame that is attached to a rigid-body such as the body-fixed frame or the sensor-fixed frame. A *global frame* (denoted by F_G) is a frame that is used as a reference frame such as the local tangent plane frame (NED frame) or a static frame on some terrain feature. Given $\mathbf{R}_L^G \in SO(3)$, the right perturbation corresponds to a perturbation of the local frame and the left perturbation corresponds to a perturbation of the global frame.

11.1 Local Perturbations

A *local perturbation* is a perturbation of the local frame defined by a rotation vector $\delta\phi_{L'L}^{L'} \in \mathbb{R}^3$. A local perturbation results in:

$$\begin{aligned} \mathbf{R}_{L'}^G &= \mathbf{R}_L^G \delta\mathbf{R}_{L'}^L = \mathbf{R}_L^G \text{Exp}(\delta\phi_{L'L}^{L'}), \\ \mathbf{R}_G^{L'} &= \delta\mathbf{R}_L^{L'} \mathbf{R}_G^L = \text{Exp}(\delta\phi_{L'L}^{L'}) \mathbf{R}_G^L = \text{Exp}(-\delta\phi_{L'L}^{L'}) \mathbf{R}_G^L, \end{aligned} \quad (380)$$

since we have $\delta\phi_{L'L}^{L'} = -\delta\phi_{L'L}^L = -\delta\phi_{L'L}^{L'}$ due to the frame invariance property. Similarly, the rotation vector associated with the local perturbation can be obtained as:

$$\begin{aligned} \delta\phi_{L'L}^{L'} &= \text{Log} [(\mathbf{R}_L^G)^T \mathbf{R}_{L'}^G] \\ &= -\text{Log} [\mathbf{R}_G^{L'} (\mathbf{R}_G^L)^T]. \end{aligned} \quad (381)$$

If the rotation angle is small, we have:

$$\begin{aligned} \delta\mathbf{R}_{L'}^L &\approx \mathbf{I} + \left[\delta\phi_{L'L}^{L'} \right]_{\times}, \\ \delta\mathbf{R}_L^{L'} &\approx \mathbf{I} - \left[\delta\phi_{L'L}^{L'} \right]_{\times}. \end{aligned} \quad (382)$$

11.2 Global Perturbations

A *global perturbation* is perturbation of the global frame defined by a rotation vector $\delta\phi_{G'G}^{G'} \in \mathbb{R}^3$. A global perturbation results in:

$$\begin{aligned} \mathbf{R}_L^{G'} &= \delta\mathbf{R}_G^{G'} \mathbf{R}_L^G = \text{Exp}(-\delta\phi_{G'G}^{G'}) \mathbf{R}_L^G, \\ \mathbf{R}_{G'}^L &= \mathbf{R}_G^L \delta\mathbf{R}_{G'}^G = \mathbf{R}_G^L \text{Exp}(\delta\phi_{G'G}^{G'}). \end{aligned} \quad (383)$$

The rotation vector associated with the global perturbation can be obtained as:

$$\begin{aligned} \delta\phi_{G'G}^{G'} &= -\text{Log} \left[\mathbf{R}_L^{G'} (\mathbf{R}_L^G)^T \right] \\ &= \text{Log} \left[(\mathbf{R}_G^L)^T \mathbf{R}_{G'}^L \right]. \end{aligned} \quad (384)$$

12 Jacobian Definitions on $SO(3)$

For reference, recall the classical derivative in Euclidean space. Let $\mathbf{f} : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a vector-valued function. The derivative at $\mathbf{x} \in \mathbb{R}^m$ is an $n \times m$ Jacobian matrix. The j -th column is obtained by perturbing the j -th component of \mathbf{x} :

$$\text{col}_j \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}) \right) = \lim_{h \rightarrow 0} \frac{\mathbf{f}(\mathbf{x} + h\mathbf{i}_j) - \mathbf{f}(\mathbf{x})}{h} \in \mathbb{R}^n, \quad (385)$$

where $\mathbf{i}_j \in \mathbb{R}^m$ is a unit column vector whose entries are all zero except the j -th entry, which is 1. Therefore, the Jacobian is:

$$\mathbf{J}_x^{\mathbf{f}} := \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}) = \left[\text{col}_1 \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}) \right) \quad \cdots \quad \text{col}_m \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}) \right) \right] \in \mathbb{R}^{n \times m}, \quad j = 1, \dots, m. \quad (386)$$

The first-order Taylor approximation is:

$$\mathbf{f}(\mathbf{x} + \delta\mathbf{x}) \approx \mathbf{f}(\mathbf{x}) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}) \delta\mathbf{x}. \quad (387)$$

This column-by-column formulation extends naturally to functions involving $SO(3)$. The classical definition of derivatives can still be used with the rotation composition and difference operations defined in Sections 10.6.1 and 10.6.2 as the addition and subtraction operations.

12.1 Functions from $SO(3)$ to \mathbb{R}^n

Let $\mathbf{f} : SO(3) \rightarrow \mathbb{R}^n$ and let $\mathbf{R} \in SO(3)$. Let $\phi = \mathbf{u}\theta \in \mathbb{R}^3$ be a rotation vector locally representing \mathbf{R} such that $\mathbf{R} = \text{Exp}(\phi)$. The Jacobian is an $n \times 3$ matrix whose j -th column is:

$$\text{col}_j \left(\frac{\partial \mathbf{f}(\mathbf{R})}{\partial \phi} \right) := \lim_{h \rightarrow 0} \frac{\mathbf{f}(\mathbf{R} \text{Exp}(h\mathbf{i}_j)) - \mathbf{f}(\mathbf{R})}{h} \in \mathbb{R}^n, \quad (388)$$

where $h\mathbf{i}_j$ perturbs the j -th component of ϕ . Therefore, the Jacobian is:

$$\mathbf{J}_\phi^{\mathbf{f}} := \frac{\partial \mathbf{f}(\mathbf{R})}{\partial \phi} = \left[\text{col}_1 \left(\frac{\partial \mathbf{f}(\mathbf{R})}{\partial \phi} \right) \quad \text{col}_2 \left(\frac{\partial \mathbf{f}(\mathbf{R})}{\partial \phi} \right) \quad \text{col}_3 \left(\frac{\partial \mathbf{f}(\mathbf{R})}{\partial \phi} \right) \right] \in \mathbb{R}^{n \times 3}. \quad (389)$$

The first-order approximation is:

$$\mathbf{f}(\mathbf{R} \text{Exp}(\delta\phi)) \approx \mathbf{f}(\mathbf{R}) + \frac{\partial \mathbf{f}(\mathbf{R})}{\partial \phi} \delta\phi. \quad (390)$$

12.2 Functions from \mathbb{R}^m to $SO(3)$

Let $\mathbf{f} : \mathbb{R}^m \rightarrow SO(3)$ and let $\mathbf{x} \in \mathbb{R}^m$. Since the output is a rotation matrix, the difference between two outputs must be measured using the rotation difference (Section 10.6.2). The Jacobian is a $3 \times m$ matrix whose j -th column is:

$$\text{col}_j \left(\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} \right) := \lim_{h \rightarrow 0} \frac{\text{Log}(\mathbf{f}(\mathbf{x})^T \mathbf{f}(\mathbf{x} + h\mathbf{i}_j))}{h} \in \mathbb{R}^3. \quad (391)$$

Therefore, the Jacobian is:

$$\mathbf{J}_{\mathbf{x}}^{\mathbf{f}} := \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \left[\text{col}_1 \left(\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} \right) \quad \cdots \quad \text{col}_m \left(\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} \right) \right] \in \mathbb{R}^{3 \times m}. \quad (392)$$

The first-order approximation is:

$$\mathbf{f}(\mathbf{x} + \delta \mathbf{x}) \approx \mathbf{f}(\mathbf{x}) \text{Exp} \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \delta \mathbf{x} \right). \quad (393)$$

12.3 Functions from $SO(3)$ to $SO(3)$

Let $\mathbf{f} : SO(3) \rightarrow SO(3)$ and let $\mathbf{R} \in SO(3)$. Both input perturbation and output difference require $SO(3)$ operations. The Jacobian is a 3×3 matrix whose j -th column is:

$$\text{col}_j \left(\frac{\partial \mathbf{f}(\mathbf{R})}{\partial \phi} \right) := \lim_{h \rightarrow 0} \frac{\text{Log}[\mathbf{f}(\mathbf{R})^T \mathbf{f}(\mathbf{R} \text{Exp}(h\mathbf{i}_j))]}{h} \in \mathbb{R}^3. \quad (394)$$

Therefore, the Jacobian is:

$$\mathbf{J}_{\phi}^{\mathbf{f}} := \frac{\partial \mathbf{f}(\mathbf{R})}{\partial \phi} = \left[\text{col}_1 \left(\frac{\partial \mathbf{f}(\mathbf{R})}{\partial \phi} \right) \quad \text{col}_2 \left(\frac{\partial \mathbf{f}(\mathbf{R})}{\partial \phi} \right) \quad \text{col}_3 \left(\frac{\partial \mathbf{f}(\mathbf{R})}{\partial \phi} \right) \right] \in \mathbb{R}^{3 \times 3}. \quad (395)$$

The first-order approximation is:

$$\mathbf{f}(\mathbf{R} \text{Exp}(\delta \phi)) \approx \mathbf{f}(\mathbf{R}) \text{Exp} \left(\frac{\partial \mathbf{f}(\mathbf{R})}{\partial \phi} \delta \phi \right). \quad (396)$$

12.4 The Left and Right Jacobians of $SO(3)$

The left and right Jacobians of $SO(3)$ play an important role allowing us to relate a change in $SO(3)$ to a change in $\mathfrak{so}(3)$.

12.4.1 Baker–Campbell–Hausdorff Formula

Let $\phi_1, \phi_2 \in \mathbb{R}^3$ be rotation vectors and let $\mathbf{R}_1, \mathbf{R}_2 \in SO(3)$ be their associated rotation matrices, respectively. Let $\mathbf{R}_3 \in SO(3)$ be a rotation obtained by the composition $\mathbf{R}_1 \mathbf{R}_2$. Since $SO(3)$ is not a vector space, we have:

$$\mathbf{R}_3 = \mathbf{R}_1 \mathbf{R}_2 = \text{Exp}(\phi_1) \text{Exp}(\phi_2) \neq \text{Exp}(\phi_1 + \phi_2). \quad (397)$$

Since $\mathbf{R}_3 \in SO(3)$, there exists an associated rotation vector $\phi_3 \in \mathbb{R}^3$ such that $\text{Exp}(\phi_3) = \text{Exp}(\phi_1) \text{Exp}(\phi_2)$. The *Baker–Campbell–Hausdorff formula* (BCH) expresses ϕ_3 in terms of ϕ_1 and ϕ_2 using only Lie algebra operations:

$$\phi_3 = \phi_1 + \phi_2 + \frac{1}{2}[\phi_1, \phi_2] + \frac{1}{12}[\phi_1, [\phi_1, \phi_2]] + \frac{1}{12}[\phi_2, [\phi_2, \phi_1]] + \dots, \quad (398)$$

where $\Phi_i := [\phi_i]_{\times} \in \mathfrak{so}(3)$ for $i = 1, 2, 3$, and $[\cdot, \cdot]$ is the Lie bracket of $\mathfrak{so}(3)$. Using the adjoint matrix of $\mathfrak{so}(3)$, it can be equivalently written as:

$$\phi_3 = \phi_1 + \phi_2 + \frac{1}{2} [\phi_1]_{\times} \phi_2 + \frac{1}{12} [\phi_1]_{\times}^2 \phi_2 + \frac{1}{12} [\phi_2]_{\times}^2 \phi_1 + \dots \quad (399)$$

Therefore, BCH formula defines how the rotation composition $\mathbf{R}_1 \mathbf{R}_2$ will look like in the tangent space. The full BCH formula involves infinitely many nested Lie brackets and is rarely used directly. Its practical value lies in the first-order truncation when one of the two rotation vectors is small.

12.4.2 Right Jacobian of $SO(3)$

If $\delta\phi_2$ is sufficiently small, we can discard all terms of order $\mathcal{O}(\delta\phi_2^2)$ and higher:

$$\begin{aligned} \phi_3 &\approx \phi_1 + \delta\phi_2 + \frac{1}{2} [\phi_1]_{\times} \delta\phi_2 + \frac{1}{12} [\phi_1]_{\times}^2 \delta\phi_2 - \frac{1}{720} [\phi_1]_{\times}^4 \delta\phi_2 + \dots \\ &= \phi_1 + \left(\mathbf{I} + \frac{1}{2} [\phi_1]_{\times} + \frac{1}{12} [\phi_1]_{\times}^2 - \frac{1}{720} [\phi_1]_{\times}^4 + \dots \right) \delta\phi_2 \\ &= \phi_1 + \left[\sum_{n=0}^{\infty} \frac{B_n}{n!} ([\phi_1]_{\times})^n \right] \delta\phi_2, \end{aligned} \quad (400)$$

where B_n are the *Bernoulli numbers* ($B_0 = 1, B_1 = \frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, \dots$). The series above is infinite in general, but for $\mathfrak{so}(3)$, it converges to a finite expression. Any skew-symmetric matrix $[\phi]_{\times}$ with $\phi = \mathbf{u}\theta$ satisfies the cubic identity $[\phi]_{\times}^3 = -\theta^2 [\phi]_{\times}$. Therefore, all higher powers reduce to linear combinations of $[\phi]_{\times}$ and $[\phi]_{\times}^2$:

$$[\phi]_{\times}^{2k+1} = (-1)^k \theta^{2k} [\phi]_{\times}, \quad [\phi]_{\times}^{2k} = (-1)^{k-1} \theta^{2(k-1)} [\phi]_{\times}^2, \quad k \geq 1. \quad (401)$$

Therefore, (400) can be rewritten as:

$$\phi_3 \approx \phi_1 + \left[\mathbf{I} + \underbrace{\left(\sum_{k=0}^{\infty} \frac{B_{2k+1}}{(2k+1)!} (-1)^k \theta^{2k} \right)}_{\alpha} [\phi]_{\times} + \underbrace{\left(\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} (-1)^{k-1} \theta^{2(k-1)} \right)}_{\beta} [\phi]_{\times}^2 \right] \delta\phi_2 \quad (402)$$

The odd-indexed Bernoulli numbers vanish for $k \geq 1$ (i.e., $B_3 = B_5 = \dots = 0$), so the coefficient α reduces to $\frac{1}{2}$. For the coefficient β , the Bernoulli-number generating function gives $\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n$, from which the even-indexed sum evaluates to:

$$\beta = \frac{1}{\theta^2} - \frac{1 + \cos \theta}{2\theta \sin \theta}. \quad (403)$$

Therefore, this yields:

$$\phi_3 \approx \phi_1 + \underbrace{\left[\mathbf{I} + \frac{1}{2} [\phi_1]_{\times} + \left(\frac{1}{\theta^2} - \frac{1 + \cos \theta}{2\theta \sin \theta} \right) [\phi_1]_{\times}^2 \right]}_{\mathbf{J}_r^{-1}(\phi_1)} \delta\phi_2, \quad (404)$$

where we defined the inverse of the *right Jacobian of $SO(3)$* as:

$$\mathbf{J}_r^{-1}(\phi) := \mathbf{I} + \frac{1}{2} [\phi]_{\times} + \left(\frac{1}{\theta^2} - \frac{1 + \cos \theta}{2\theta \sin \theta} \right) [\phi]_{\times}^2. \quad (405)$$

The right Jacobian $\mathbf{J}_r(\phi)$ is obtained by inverting (405):

$$\mathbf{J}_r(\phi) := \mathbf{I} - \frac{1 - \cos \theta}{\theta^2} [\phi]_{\times} + \frac{\theta - \sin \theta}{\theta^3} [\phi]_{\times}^2. \quad (406)$$

12.4.3 Left Jacobian of $SO(3)$

Similarly, if $\delta\phi_1$ is sufficiently small, we discard all terms of order $\mathcal{O}(\delta\phi_1^2)$ and higher. The degree 2 BCH term $\frac{1}{2}[\delta\phi]_{\times}\phi = -\frac{1}{2}[\phi]_{\times}\delta\phi$ now carries a sign flip relative to the right Jacobian case:

$$\begin{aligned}
\phi_3 &\approx \delta\phi_1 + \phi_2 + \frac{1}{2}[\delta\phi_1]_{\times}\phi_2 + \frac{1}{12}[\phi_2]_{\times}^2\delta\phi_1 - \frac{1}{720}[\phi_2]_{\times}^4\delta\phi_1 + \dots \\
&= \delta\phi_1 + \phi_2 - \frac{1}{2}[\phi_2]_{\times}\delta\phi_1 + \frac{1}{12}[\phi_2]_{\times}^2\delta\phi_1 - \frac{1}{720}[\phi_2]_{\times}^4\delta\phi_1 + \dots \\
&= \left(\mathbf{I} - \frac{1}{2}[\phi_2]_{\times} + \frac{1}{12}[\phi_2]_{\times}^2 - \frac{1}{720}[\phi_2]_{\times}^4 + \dots \right) \delta\phi_1 + \phi_2 + \\
&= \left[\sum_{n=0}^{\infty} \frac{(-1)^n B_n}{n!} ([\phi_2]_{\times})^n \right] \delta\phi_1 + \phi_2.
\end{aligned} \tag{407}$$

Comparing with the right Jacobian series (400), the only difference is the sign of the odd-powered terms (the $(-1)^n$ factor), which is equivalent to replacing ϕ with $-\phi$. Therefore, the inverse of the *left Jacobian of $SO(3)$* is defined as:

$$\mathbf{J}_l^{-1}(\phi) := \mathbf{I} - \frac{1}{2}[\phi]_{\times} + \left(\frac{1}{\theta^2} - \frac{1 + \cos\theta}{2\theta \sin\theta} \right) [\phi]_{\times}^2 = \mathbf{J}_r^{-1}(-\phi). \tag{408}$$

The left Jacobian $\mathbf{J}_l(\phi)$ is obtained by inverting (408):

$$\mathbf{J}_l(\phi) := \mathbf{I} + \frac{1 - \cos\theta}{\theta^2}[\phi]_{\times} + \frac{\theta - \sin\theta}{\theta^3}[\phi]_{\times}^2 = \mathbf{J}_r(-\phi). \tag{409}$$

12.4.4 Properties of the Left and Right Jacobians

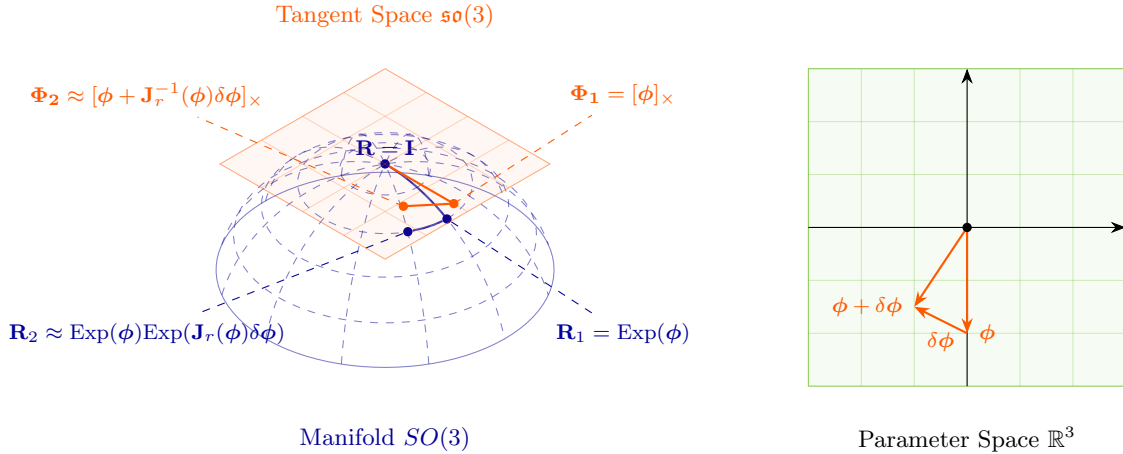


Figure 9: The right Jacobian of $SO(3)$ maps $\phi + \delta\phi$ in the parameter space into a proper rotation matrix in $SO(3)$

The left and right Jacobians immediately relate to the right and left perturbations. Let F_A and F_B be frames and let $\mathbf{R}_B^A \in SO(3)$. Let $\delta\phi_{AA'}^A \in \mathbb{R}^3$ and $\delta\phi_{B'B}^B \in \mathbb{R}^3$ be the left and right perturbation vectors such that:

$$\mathbf{R}_B^{A'} = \text{Exp}(\delta\phi_{AA'}^A) \mathbf{R}_B^A, \quad \mathbf{R}_{B'}^A = \mathbf{R}_B^A \text{Exp}(\delta\phi_{B'B}^B). \tag{410}$$

Using the left and right Jacobians, we have:

$$\phi_{BA'}^B \approx \phi_{BA}^B + \mathbf{J}_l^{-1}(\phi_{BA}^B) \delta\phi_{AA'}^A, \quad \phi_{B'A}^B \approx \phi_{BA}^B + \mathbf{J}_r^{-1}(\phi_{BA}^B) \delta\phi_{B'B}^B. \tag{411}$$

Therefore, we get:

$$\begin{aligned} \text{Exp}(\delta\phi_{AA'}^A)\text{Exp}(\phi_{BA}^B) &\approx \text{Exp}(\phi_{BA}^B + \mathbf{J}_l^{-1}(\phi_{BA}^B)\delta\phi_{AA'}^A) \\ \text{Exp}(\phi_{BA}^B)\text{Exp}(\phi_{B'B}^{B'}) &\approx \text{Exp}(\phi_{BA}^B + \mathbf{J}_r^{-1}(\phi_{BA}^B)\delta\phi_{B'B}^{B'}). \end{aligned} \quad (412)$$

Conversely, let ϕ_{BA}^B be perturbed in the tangent space by a perturbation vector $\delta\phi \in \mathbb{R}^3$. From (412), we have:

$$\begin{aligned} \text{Exp}(\phi_{BA}^B + \delta\phi) &\approx \text{Exp}(\mathbf{J}_l(\phi_{BA}^B)\delta\phi)\text{Exp}(\phi_{BA}^B), \\ &\approx \text{Exp}(\phi_{BA}^B)\text{Exp}(\mathbf{J}_r(\phi_{BA}^B)\delta\phi). \end{aligned} \quad (413)$$

Since $\delta\phi$ is a perturbation vector in the tangent space, it did not have any meaning in terms of frame-to-frame relationship. However, it once the mapping is done using (412), $\delta\phi \equiv \delta\phi_{AA'}^A$ if the left Jacobian is used and $\delta\phi \equiv \delta\phi_{B'B}^{B'}$ if the right Jacobian is used.

Additionally, the left and right Jacobians are related by:

$$\begin{aligned} \mathbf{J}_l(\phi) &= \mathbf{J}_r(-\phi) = \mathbf{J}_r(\phi)^T, \\ \mathbf{J}_l(\phi) &= \mathbf{R}\mathbf{J}_r(\phi), \quad (\mathbf{R} = \text{Exp}(\phi)). \end{aligned} \quad (414)$$

The transpose relation $\mathbf{J}_l = \mathbf{J}_r^T$ follows from comparing the closed-form expressions: the $[\phi]_\times$ term flips sign under transposition (since $[\phi]_\times^T = -[\phi]_\times$), while the \mathbf{I} and $[\phi]_\times^2$ terms are symmetric.

At $\phi = \mathbf{0}$, the right Jacobian reduces to $\mathbf{J}_r(\mathbf{0}) = \mathbf{I}$, recovering the identity: an infinitesimal perturbation about the nominal rotation maps directly to the tangent space without distortion. For finite ϕ , \mathbf{J}_r accounts for the curvature of $SO(3)$. This Jacobian is essential in state estimation for propagating covariance in error-state formulations.

13 Jacobians of Rotations

13.1 Jacobian of Rotation Function with Respect to the Rotated Vector

Let $\mathbf{R} \in SO(3)$ and $\mathbf{x} \in \mathbb{R}^3$ be a component vector in a frame. We define the function $\mathbf{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ as:

$$\mathbf{f}(\mathbf{x}) := \mathbf{R}\mathbf{x}. \quad (415)$$

The Jacobian of \mathbf{f} with respect to \mathbf{x} is a 3×3 matrix. Using (385), the j -th column is obtained by perturbing the j -th component of \mathbf{x} :

$$\begin{aligned} \text{col}_j\left(\frac{\partial(\mathbf{R}\mathbf{x})}{\partial\mathbf{x}}\right) &= \lim_{h \rightarrow 0} \frac{\mathbf{R}(\mathbf{x} + h\mathbf{i}_j) - \mathbf{R}\mathbf{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\mathbf{R}h\mathbf{i}_j}{h} \\ &= \mathbf{R}\mathbf{i}_j. \end{aligned} \quad (416)$$

Therefore, the Jacobian is:

$$\mathbf{J}_x^{\mathbf{R}\mathbf{x}} := \frac{\partial(\mathbf{R}\mathbf{x})}{\partial\mathbf{x}} = \mathbf{R} \in \mathbb{R}^{3 \times 3}. \quad (417)$$

13.2 Jacobian of Rotation Matrix with Respect to the Rotation Vector

Let $\mathbf{R} \in SO(3)$ be a rotation parameterized by a rotation vector $\phi \in \mathbb{R}^3$ such that $\mathbf{R} = \text{Exp}(\phi)$. The function $\mathbf{f} : \mathbb{R}^3 \rightarrow SO(3)$ is defined as:

$$\mathbf{f}(\phi) := \text{Exp}(\phi). \quad (418)$$

The Jacobian \mathbf{f} with respect to ϕ is a 3×3 matrix. Using (391) and (413), the j -th column is obtained by perturbing the j -th component of ϕ :

$$\begin{aligned}
\text{col}_j \left(\frac{\partial (\text{Exp}(\phi))}{\partial \phi} \right) &= \lim_{h \rightarrow 0} \frac{\text{Log} \left((\text{Exp}(\phi))^T \text{Exp}(\phi + h \mathbf{i}_j) \right)}{h} \\
&\approx \lim_{h \rightarrow 0} \frac{\text{Log} \left((\text{Exp}(\phi))^T \text{Exp}(\phi) \text{Exp}(\mathbf{J}_r(\phi) h \mathbf{i}_j) \right)}{h} \\
&= \lim_{h \rightarrow 0} \frac{\text{Log}(\text{Exp}(\mathbf{J}_r(\phi) h \mathbf{i}_j))}{h} \\
&= \lim_{h \rightarrow 0} \frac{\mathbf{J}_r(\phi) h \mathbf{i}_j}{h} \\
&= \mathbf{J}_r(\phi) \mathbf{i}_j.
\end{aligned} \tag{419}$$

Therefore, the right Jacobian of $SO(3)$ is exactly the derivative of the rotation matrix with respect to the rotation vector:

$$\boxed{\mathbf{J}_\phi^{\mathbf{R}} := \mathbf{J}_r = \frac{\partial \text{Exp}(\phi)}{\partial \phi} \in \mathbb{R}^{3 \times 3}.} \tag{420}$$

13.3 Jacobian of Rotation Function with Respect to the Rotation Vector

Let $\mathbf{R} \in SO(3)$ be a rotation parameterized by the rotation vector $\phi \in \mathbb{R}^3$ such that $\mathbf{R} = \text{Exp}(\phi)$. Let $\mathbf{x} \in \mathbb{R}^3$ be a component vector in a frame. We define the function $\mathbf{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$:

$$\mathbf{f}(\phi) := \text{Exp}(\phi) \mathbf{x}. \tag{421}$$

For a small $\delta\phi \in \mathbb{R}^3$, using (413), we know that:

$$\begin{aligned}
\text{Exp}(\phi + \delta\phi) \mathbf{x} &\approx \text{Exp}(\phi) \text{Exp}(\mathbf{J}_r(\phi) \delta\phi) \mathbf{x} \\
&\approx \text{Exp}(\phi) (\mathbf{I} + [\mathbf{J}_r(\phi) \delta\phi]_\times) \mathbf{x} \\
&= \mathbf{R} \mathbf{x} + \mathbf{R} [\mathbf{x}]_\times \mathbf{J}_r(\phi) \delta\phi.
\end{aligned} \tag{422}$$

The Jacobian of \mathbf{f} with respect to ϕ is a 3×3 matrix. Using above and (385), the j -th column is obtained by perturbing the j -th component of ϕ :

$$\begin{aligned}
\text{col}_j \left(\frac{\partial (\text{Exp}(\phi) \mathbf{x})}{\partial \phi} \right) &= \lim_{h \rightarrow 0} \frac{\text{Exp}(\phi + h \mathbf{i}_j) \mathbf{x} - \text{Exp}(\phi) \mathbf{x}}{h} \\
&\approx \lim_{h \rightarrow 0} \frac{\mathbf{R} \mathbf{x} + \mathbf{R} [\mathbf{x}]_\times \mathbf{J}_r(\phi) h \mathbf{i}_j - \mathbf{R} \mathbf{x}}{h} \\
&= \mathbf{R} [\mathbf{x}]_\times \mathbf{J}_r(\phi).
\end{aligned} \tag{423}$$

Therefore, the Jacobian is:

$$\boxed{\mathbf{J}_\phi^{\mathbf{R} \mathbf{x}} := \frac{\partial (\mathbf{R} \mathbf{x})}{\partial \phi} = -\mathbf{R} [\mathbf{x}]_\times \mathbf{J}_r(\phi) \in \mathbb{R}^{3 \times 3}.} \tag{424}$$

For small $\phi = \delta\phi$, $\mathbf{J}_r \approx \mathbf{I}$ and this reduces to:

$$\mathbf{J}_\phi^{\mathbf{R} \mathbf{x}} \approx -\mathbf{R} [\mathbf{x}]_\times. \tag{425}$$

This small-angle limit is the Jacobian most commonly encountered in state estimation, where the perturbation $\delta\phi_{B/B}^B$ about the current estimate is small by assumption.

13.4 Jacobian of Rotation Function with Respect to the Quaternion

Let $\underline{q}_B^A \in \mathbb{H}$ be a unit quaternion representing the rotation from frame F_B to frame F_A . Let $\mathbf{q}_B^A = [q_0, q_1, q_2, q_3]^T \in \mathbb{R}^4$ be the Hamiltonian quaternion vector and let $\mathbf{x}^B \in \mathbb{R}^3$. Using (199), we define the rotation function $\mathbf{f} : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ as:

$$\mathbf{f}(\mathbf{q}_B^A) := (q_0^2 - \mathbf{q}_v^T \mathbf{q}_v) \mathbf{x}^B + 2(\mathbf{q}_v^T \mathbf{x}^B) \mathbf{q}_v + 2q_0(\mathbf{q}_v \times \mathbf{x}^B). \quad (426)$$

The Jacobian of \mathbf{f} with respect to the quaternion \mathbf{q} is a 3×4 matrix. The j 'th column can be computed by differentiating directly with respect to the q_j parameter:

$$\begin{aligned} \frac{\partial \mathbf{f}(\mathbf{q}_B^A)}{\partial q_0} &= 2q_0 \mathbf{x}^B + 2(\mathbf{q}_v \times \mathbf{x}^B) \\ &= 2(q_0 \mathbf{x}^B + \mathbf{q}_v \times \mathbf{x}^B), \\ \frac{\partial \mathbf{f}(\mathbf{q}_B^A)}{\partial \mathbf{q}_v} &= -2\mathbf{q}_v^T \mathbf{x}^B + 2(\mathbf{x}^B)^T \mathbf{q}_v + 2(\mathbf{q}_v^T \mathbf{x}^B) - 2q_0 [\mathbf{x}^B]_{\times} \\ &= 2\left(\mathbf{q}_v^T \mathbf{x}^B \mathbf{I} + \mathbf{q}_v (\mathbf{x}^B)^T - \mathbf{x}^B \mathbf{q}_v^T - q_0 [\mathbf{x}^B]_{\times}\right). \end{aligned} \quad (427)$$

Therefore, the Jacobian is:

$$\mathbf{J}_{\mathbf{q}}^{\mathbf{R}^{\mathbf{x}}} := 2 \begin{bmatrix} q_0 \mathbf{x}^B + \mathbf{q}_v \times \mathbf{x}^B & \mathbf{q}_v^T \mathbf{x}^B \mathbf{I} + \mathbf{q}_v (\mathbf{x}^B)^T - \mathbf{x}^B \mathbf{q}_v^T - q_0 [\mathbf{x}^B]_{\times} \end{bmatrix} \in \mathbb{R}^{3 \times 4}. \quad (428)$$

13.5 Jacobian of Rotation Matrix Composition

Let $\mathbf{R}_B^A, \mathbf{R}_C^B \in SO(3)$ be rotations parameterized by the rotation vectors $\phi_{BA}^B, \phi_{CB}^C \in \mathbb{R}^3$ such that $\mathbf{R}_B^A = \text{Exp}(\phi_{BA}^B)$ and $\mathbf{R}_C^B = \text{Exp}(\phi_{CB}^C)$. Define $\mathbf{f} : SO(3) \rightarrow SO(3)$ as:

$$\mathbf{f}(\mathbf{R}_B^A) := \mathbf{R}_B^A \mathbf{R}_C^B. \quad (429)$$

The right perturbation of \mathbf{R}_B^A by $\delta \mathbf{R} \in SO(3)$ results in $\mathbf{f}(\mathbf{R}_B^A \delta \mathbf{R}) = (\mathbf{R}_B^A \delta \mathbf{R}) \mathbf{R}_C^B$. Let $\delta \phi \in \mathbb{R}^3$ be the corresponding perturbation vector such that $\delta \mathbf{R} = \text{Exp}(\delta \phi)$. The Jacobian of \mathbf{f} with respect to ϕ_{BA}^B is a 3×3 matrix. Using (394), the j -th column is obtained by perturbing the j -th component of ϕ_{BA}^B :

$$\begin{aligned} \text{col}_j \left(\frac{\partial (\mathbf{R}_B^A \mathbf{R}_C^B)}{\partial \phi_{BA}^B} \right) &= \lim_{h \rightarrow 0} \frac{\text{Log} \left((\mathbf{R}_B^A \mathbf{R}_C^B)^T (\mathbf{R}_B^A \text{Exp}(h \mathbf{i}_j) \mathbf{R}_C^B) \right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\text{Log} \left((\mathbf{R}_C^B)^T \text{Exp}(h \mathbf{i}_j) \mathbf{R}_C^B \right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\text{Log} \left(\text{Exp} \left((\mathbf{R}_C^B)^T h \mathbf{i}_j \right) \right)}{h} \\ &= (\mathbf{R}_C^B)^T \mathbf{i}_j. \end{aligned} \quad (430)$$

Defining $\mathbf{R}_1 := \mathbf{R}_B^A$, $\mathbf{R}_2 := \mathbf{R}_C^B$, and $\phi_1 := \phi_{BA}^B$, the Jacobian is:

$$\mathbf{J}_{\phi_1}^{\mathbf{R}_1 \mathbf{R}_2} := (\mathbf{R}_2)^T \in \mathbb{R}^{3 \times 3}. \quad (431)$$

Similarly, the Jacobian with respect to ϕ_{CB}^C can be obtained by defining $\mathbf{f} : SO(3) \rightarrow SO(3)$ as:

$$\mathbf{f}(\mathbf{R}_C^B) := \mathbf{R}_B^A \mathbf{R}_C^B. \quad (432)$$

The right perturbation of \mathbf{R}_C^B by $\delta\mathbf{R}$ results in $\mathbf{f}(\mathbf{R}_C^B\delta\mathbf{R}) = \mathbf{R}_B^A(\mathbf{R}_C^B\delta\mathbf{R})$. The Jacobian of $\mathbf{f}(\mathbf{R}_C^B)$ with respect to ϕ_{CB}^C is 3×3 matrix. Using (394), the j -th column is obtained by perturbing the j -th component of ϕ_{CB}^C :

$$\begin{aligned} \text{col}_j\left(\frac{\partial(\mathbf{R}_B^A\mathbf{R}_C^B)}{\phi_{CB}^C}\right) &= \lim_{h \rightarrow 0} \frac{\text{Log}\left(\left(\mathbf{R}_B^A\mathbf{R}_C^B\right)^T \left(\mathbf{R}_B^A\mathbf{R}_C^B \text{Exp}(h\mathbf{i}_j)\right)\right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\text{Log}(\text{Exp}(h\mathbf{i}_j))}{h} \\ &= \lim_{h \rightarrow 0} \\ &= \mathbf{i}_j. \end{aligned} \tag{433}$$

Defining $\phi_2 := \phi_{CB}^C$, the Jacobian is:

$$\boxed{\mathbf{J}_{\phi_2}^{\mathbf{R}_1\mathbf{R}_2} = \mathbf{I} \in \mathbb{R}^{3 \times 3}} \tag{434}$$

13.6 Jacobian of Quaternion Composition

Let $\mathbf{q}_B^A, \mathbf{q}_C^B \in \mathbb{R}^4$ be rotations parameterized by Hamilton quaternions. Let $\phi_{BA}^B, \phi_{CB}^C \in \mathbb{R}^3$ be the associated rotation vectors. The Jacobian of quaternion composition with respect to the rotation vectors are identical to the results from Section 13.5. Defining $\mathbf{q}_1 := \mathbf{q}_B^A$, $\mathbf{q}_2 := \mathbf{q}_C^B$, $\phi_1 := \phi_{BA}^B$, and $\phi_2 := \phi_{CB}^C$, we get:

$$\mathbf{J}_{\phi_1}^{\mathbf{q}_1 \otimes \mathbf{q}_2} := (\mathbf{R}_2)^T, \quad \mathbf{J}_{\phi_2}^{\mathbf{q}_1 \otimes \mathbf{q}_2} := \mathbf{I}. \tag{435}$$

14 Integration of Rotation Rates

We have shown in Parts IV and V that rotation evolve in time according to the kinematic equation:

$$\dot{\mathbf{R}}_B^A(t) = \mathbf{R}_B^A(t)[\boldsymbol{\omega}_{BA}^B(t)]_{\times}, \tag{436}$$

where $\boldsymbol{\omega}_{BA}^B$ is the angular velocity of frame F_B relative to frame F_A , resolved in frame F_B . Numerically propagating this equation requires discrete-time integration schemes that keep the result on $SO(3)$. Naive Euler integration ($\mathbf{R}_{Bk+1}^A = \mathbf{R}_{Bk}^A + \mathbf{R}_{Bk}^A\Delta t$) pushes the result off the manifold. Instead, integration is performed through the exponential map.

14.1 Zeroth-Order Integration

14.1.1 Forward Integration

The simplest scheme assumes the angular velocity $\boldsymbol{\omega}_{BA}^B$ is constant over the time step $[t_k, t_{k+1}]$, with $\Delta t = t_{k+1} - t_k$. From the closed-form solution of the kinematic equation:

$$\mathbf{R}_B^A(t_{k+1}) = \mathbf{R}_B^A(t_k) \text{Exp}(\boldsymbol{\omega}_{BA}^B(t_k)\Delta t), \tag{437}$$

which is called the *zeroth-order forward integration*. This is equivalent to a single rotation by angle $\|\boldsymbol{\omega}_{BA}^B(t_k)\|\Delta t$ about axis $\boldsymbol{\omega}_{BA}^B(t_k)/\|\boldsymbol{\omega}_{BA}^B(t_k)\|$. The result lies exactly on $SO(3)$ by construction, since $\text{Exp}(\cdot)$ always returns a valid rotation matrix.

Equivalently, using the left form of the kinematic equation with $\boldsymbol{\omega}_{BA}^A = \mathbf{R}_B^A\boldsymbol{\omega}_{BA}^B$ (the angular velocity resolved in F_A via the adjoint):

$$\mathbf{R}_B^A(t_{k+1}) = \text{Exp}(\boldsymbol{\omega}_{BA}^A(t_k)\Delta t) \mathbf{R}_B^A(t_k). \tag{438}$$

14.1.2 Backward and Midward Integrations

In forward integration, we considered that the constant angular velocity over the period Δt corresponds to $\omega_{BA}^B(t_k)$. However, we can also consider $\omega_{BA}^B(t_{k+1})$ or the median rate $(\omega_{BA}^B(t_k) + \omega_{BA}^B(t_{k+1}))/2$ to be constant over Δt . These are called zeroth-order *backward* and *midward* integrations, respectively.

14.2 First-Order Integration

For improved accuracy, assume that the angular velocity varies linearly over the time step:

$$\dot{\omega}_{BA}^B = \frac{\omega_{BA}^B(t_{k+1}) - \omega_{BA}^B(t_k)}{\Delta t}. \quad (439)$$

First-order approximation becomes:

$$\omega_{BA}^B(t) \approx \omega_{BA}^B(t_k) + \frac{\omega_{BA}^B(t_{k+1}) - \omega_{BA}^B(t_k)}{\Delta t}(t - t_k), \quad t \in [t_k, t_{k+1}]. \quad (440)$$

Hence, the update becomes:

$$\mathbf{R}_B^A(t_{k+1}) = \mathbf{R}_B^A(t_k) \text{Exp} \left(\frac{\omega_{BA}^B(t_k) + \omega_{BA}^B(t_{k+1})}{2} \Delta t \right). \quad (441)$$

A more accurate first-order scheme uses the right Jacobian (Section ??) to account for the curvature of $SO(3)$:

$$\phi_{BA}^B(t_{k+1}) = \omega_{BA}^B(t_k) \Delta t, \quad (442)$$

$$\mathbf{R}_B^A(t_{k+1}) = \mathbf{R}_{Bk}^A \text{Exp}(\phi_{BA}^B(t_k) + \frac{1}{2} \mathbf{J}_r(\phi_{BA}^B(t_k)) (\omega_{BA}^B(t_{k+1}) - \omega_{BA}^B(t_k)) \Delta t). \quad (443)$$

VII Kinematics

In this part, we assemble the continuous-time and discrete-time kinematic equations of motion that govern rigid bodies, drawing on the differential tools discussed in previous section. The continuous-time equations are the ODEs satisfied by the state; the discrete-time equations are the update rules used to propagate that state over a finite time step Δt , as required by digital filters and estimators.

15 Continuous Time Systems

In continuous time, the motion of a rigid body is described by a set of ordinary differential equations relating its configuration (position and orientation) to its first- and second-order time derivatives (linear and angular velocity, linear and angular acceleration).

Let F_A be a global reference frame and let F_B be a body-fixed frame attached to the rigid body. Let B denote the origin of F_B , which is a point fixed in the rigid body, and let $\mathbf{R}_B^A(t) \in SO(3)$ denote the orientation of F_B relative to F_A .

15.1 Attitude

The orientation $\mathbf{R}_B^A(t) \in SO(3)$ of the body frame relative to the global frame evolves according to the kinematic ODE derived in Section 9.1:

$$\dot{\mathbf{R}}_B^A(t) = \mathbf{R}_B^A(t) \boldsymbol{\Omega}_{BA}^B(t) = \mathbf{R}_B^A(t) [\boldsymbol{\omega}_{BA}^B(t)]_{\times}, \quad (444)$$

where $\boldsymbol{\omega}_{BA}^B(t) \in \mathbb{R}^3$ is the angular velocity of F_B relative to F_A resolved in F_B . In applications, $\boldsymbol{\omega}_{BA}^B(t)$ is typically provided either by a rigid-body dynamics model or by a gyroscope measurement; from the standpoint of kinematics it is treated as a known input.

When orientation is parameterized by a unit quaternion $\mathbf{q}_B^A \in \mathbb{R}^4$, the kinematic ODE is derived in Section 9.2 as:

$$\dot{\mathbf{q}}_B^A(t) = \frac{1}{2} \mathbf{q}_B^A(t) \otimes {}^H \boldsymbol{\omega}_{BA}^B(t), \quad (445)$$

where ${}^H \boldsymbol{\omega}_{BA}^B(t) := [0, \boldsymbol{\omega}_{BA}^B(t)^T]^T \in \mathbb{R}^4$ is the Hamiltonian quaternion corresponding to the angular velocity. The factor $\frac{1}{2}$ arises from the double-cover of $SO(3)$ by the unit quaternions (Section 3.2.2): a rotation by angle θ corresponds to a quaternion whose vector part contains $\sin(\theta/2)$, so infinitesimal angular velocity enters with a factor of $1/2$. The ODE (445) is equivalent to (444) under the double-cover map $\mathbb{H} \rightarrow SO(3)$ and shares the same input $\boldsymbol{\omega}_{BA}^B(t)$.

Differentiating (444) once more yields the evolution of $\dot{\mathbf{R}}_B^A$, from which the *angular acceleration* $\boldsymbol{\alpha}_{BA}^B(t) := \dot{\boldsymbol{\omega}}_{BA}^B(t) \in \mathbb{R}^3$ enters as the second-order input. Angular acceleration is not an independent state variable in pure kinematics: it is the time derivative of the angular velocity input, and its specification—like that of $\boldsymbol{\omega}_{BA}^B$ itself—comes from outside the kinematic description, typically from a dynamics model or a differentiated gyroscope signal.

15.2 Position

Let x and y be points. The *position of y relative to x* is a physical vector:

$$\vec{\mathbf{p}}_{yx} \in \mathbb{V}, \quad \vec{\mathbf{p}}_{yx} = -\vec{\mathbf{p}}_{xy}. \quad (446)$$

For a third point z , vector addition gives:

$$\vec{\mathbf{p}}_{yx} = \vec{\mathbf{p}}_{yz} + \vec{\mathbf{p}}_{zx}. \quad (447)$$

Resolving this identity in a frame F_A yields the component form:

$$\mathbf{p}_{yx}^A = \mathbf{p}_{yz}^A + \mathbf{p}_{zx}^A \in \mathbb{R}^3, \quad (448)$$

where \mathbf{p}_{yx}^A is the component vector such that $\vec{\mathbf{p}}_{yx} = F_A \mathbf{p}_{yx}^A$ (Section 1.3). Components resolved in a second frame F_B relate via the rotation matrix:

$$\mathbf{p}_{yx}^B = \mathbf{R}_A^B \mathbf{p}_{yx}^A. \quad (449)$$

The position of the body origin B relative to a fixed point A in the global frame, resolved in F_A :

$$\mathbf{p}_{BA}^A(t) \in \mathbb{R}^3, \quad (450)$$

together with the orientation $\mathbf{R}_B^A(t)$, fully specifies the pose of the rigid body in F_A .

15.3 Velocity

The time derivative of a physical position vector depends on the frame in which changes are observed (Section 1.3). The *velocity of y relative to x with respect to frame F_A* is the vectrix derivative:

$${}^A \vec{\mathbf{v}}_{yx} := \widehat{{}^A \mathbf{p}}_{yx} \in \mathbb{V}. \quad (451)$$

Resolving this in the same frame F_A , that the derivative is with respect to, collapses the vectrix derivative to an ordinary component-wise time derivative:

$$\boxed{\mathbf{v}_{yx}^A := [{}^A \vec{\mathbf{v}}_{yx}]_A = {}^A \mathbf{v}_{yx}^A = \dot{\mathbf{p}}_{yx}^A \in \mathbb{R}^3,} \quad (452)$$

where the leading superscript ${}^A(\cdot)$ is omitted when the observing frame and the resolving frame coincide. Let F_B be a frame. Resolving the physical velocity vector observed in F_A to frame F_B yields:

$$[{}^A \vec{\mathbf{v}}_{yx}]_B = {}^A \mathbf{v}_{yx}^B. \quad (453)$$

Given $\mathbf{R}_A^B \in SO(3)$, we have:

$$\boxed{{}^A \mathbf{v}_{yx}^B = \mathbf{R}_A^B {}^A \mathbf{v}_{yx}^A = \mathbf{R}_A^B \dot{\mathbf{p}}_{yx}^A = \mathbf{R}_A^B \dot{\mathbf{p}}_{yx}^A.} \quad (454)$$

Velocity is registered if y moves with respect to x , or if x moves with respect to y . However, velocity is also registered if F_B rotates with respect to F_A . Suppose F_B is rotating relative to F_A with angular velocity $\vec{\omega}_{BA} \in \mathbb{V}$. The same physical position vector $\vec{\mathbf{p}}_{yx}$ has different vectrix derivatives in F_A and F_B . Using the product rule on $\vec{\mathbf{p}}_{BA} = F_B \mathbf{p}_{BA}^B$:

Suppose F_B is rotating relative to F_A with angular velocity $\vec{\omega}_{BA} \in \mathbb{V}$. The same physical position vector $\vec{\mathbf{p}}_{yx}$ has different vectrix derivatives in F_A and F_B . Using the product rule on $\vec{\mathbf{p}}_{yx} = F_B \mathbf{p}_{yx}^B$:

$$\begin{aligned} {}^A \vec{\mathbf{v}}_{yx} &= \widehat{{}^A \mathbf{p}}_{yx} = \widehat{F_B \mathbf{p}_{yx}^B} \\ &= F_B \dot{\mathbf{p}}_{yx}^B + F_B \widehat{\mathbf{p}}_{yx}^B \\ &= F_B \mathbf{v}_{yx}^B + \vec{\omega}_{BA} \times F_B \mathbf{p}_{yx}^B \\ &= {}^B \vec{\mathbf{v}}_{yx} + \vec{\omega}_{BA} \times \vec{\mathbf{p}}_{yx}, \end{aligned} \quad (455)$$

which is called the *transport theorem*. It states that the velocity observed in F_A equals the velocity observed in F_B plus a rotational correction term $\vec{\omega}_{BA} \times \vec{\mathbf{p}}_{yx}$ that accounts for the rotation of F_B relative to F_A . Resolving (455) to frame F_A yields:

$$\begin{aligned} \mathbf{v}_{yx}^A &= {}^B\mathbf{v}_{yx}^A + [\boldsymbol{\omega}_{BA}^A]_{\times} \mathbf{p}_{yx}^A \\ &= {}^B\mathbf{v}_{yx}^A + \boldsymbol{\Omega}_{BA}^A \mathbf{p}_{yx}^A. \end{aligned} \quad (456)$$

The same identity arises from differentiating the component-vector relationship $\mathbf{p}_{yx}^A = \mathbf{R}_{BA}^A \mathbf{p}_{yx}^B$ and substituting $\dot{\mathbf{R}}_B^A = \mathbf{R}_B^A \boldsymbol{\Omega}_{BA}^B$ from (444):

$$\begin{aligned} \mathbf{v}_{yx}^A &= \frac{d}{dt} (\mathbf{R}_{BA}^A \mathbf{p}_{yx}^B) = \mathbf{R}_{BA}^A \dot{\mathbf{p}}_{yx}^B + \dot{\mathbf{R}}_{BA}^A \mathbf{p}_{yx}^B \\ &= \mathbf{R}_{BA}^A \mathbf{v}_{yx}^B + \mathbf{R}_{BA}^A \boldsymbol{\Omega}_{BA}^B \mathbf{p}_{yx}^B \\ &= \mathbf{R}_{BA}^A \mathbf{v}_{yx}^B + \boldsymbol{\Omega}_{BA}^A \mathbf{p}_{yx}^A \\ &= {}^B\mathbf{v}_{yx}^A + \boldsymbol{\Omega}_{BA}^A \mathbf{p}_{yx}^A, \end{aligned} \quad (457)$$

where we used $\boldsymbol{\Omega}_{BA}^A = \mathbf{R}_B^A \boldsymbol{\Omega}_{BA}^B \mathbf{R}_A^B$ and $\mathbf{p}_{yx}^A = \mathbf{R}_B^A \mathbf{p}_{yx}^B$.

15.4 Acceleration

The *acceleration of y relative to x with respect to F_A* is obtained by differentiating the velocity in F_A a second time:

$${}^A\vec{\mathbf{a}}_{yx} := \widehat{{}^A\vec{\mathbf{v}}_{yx}} = \widehat{\vec{\mathbf{p}}_{yx}^A} \in \mathbb{V}. \quad (458)$$

When resolved in F_A itself, this is simply the second time derivative of the component vector:

$$\mathbf{a}_{yx}^A := [{}^A\vec{\mathbf{a}}_{yx}]_A = \ddot{\mathbf{p}}_{yx}^A \in \mathbb{R}^3. \quad (459)$$

Let F_B be a frame. Resolving the physical acceleration vector observed in F_A to frame F_B yields:

$$[{}^A\vec{\mathbf{a}}_{yx}]_B = {}^A\mathbf{a}_{yx}^B. \quad (460)$$

Given $\mathbf{R}_A^B \in SO(3)$, we have:

$${}^A\mathbf{a}_{yx}^B = \mathbf{R}_A^B {}^A\mathbf{a}_{yx}^A = \mathbf{R}_A^B \mathbf{a}_{yx}^A = \mathbf{R}_A^B \dot{\mathbf{v}}_{yx}^A = \mathbf{R}_A^B \dot{\mathbf{p}}_{yx}^A. \quad (461)$$

Applying the transport theorem (455) twice to $\vec{\mathbf{p}}_{yx}$ yields the acceleration of a point whose position is naturally expressed in a rotating frame F_B but whose acceleration is desired in the global frame F_A . Differentiating (455) in F_A yields:

$${}^A\vec{\mathbf{a}}_{yx} = \widehat{{}^B\vec{\mathbf{v}}_{yx}} + \widehat{\boldsymbol{\omega}_{BA}^A} \times \vec{\mathbf{p}}_{yx} + \boldsymbol{\omega}_{BA}^A \times \widehat{\vec{\mathbf{p}}_{yx}}. \quad (462)$$

The first term is itself a frame derivative and expands once more via the transport theorem:

$$\widehat{{}^B\vec{\mathbf{v}}_{yx}} = \widehat{{}^F_B\mathbf{v}_{yx}^B} \quad (463)$$

$$= {}^F_B\mathbf{v}_{yx}^B + \widehat{{}^F_B\mathbf{v}_{yx}^B} \quad (464)$$

$$= {}^F_B\dot{\mathbf{v}}_{yx}^B + \boldsymbol{\omega}_{BA}^A \times {}^F_B\mathbf{v}_{yx}^B \quad (465)$$

$$= {}^F_B\mathbf{a}_{yx}^B + \boldsymbol{\omega}_{BA}^A \times {}^B\vec{\mathbf{v}}_{yx} \quad (466)$$

$$= {}^B\vec{\mathbf{a}}_{yx} + \boldsymbol{\omega}_{BA}^A \times {}^B\vec{\mathbf{v}}_{yx}. \quad (467)$$

$$\widehat{B}^A \vec{v}_{yx} = \widehat{B}^B \vec{v}_{yx} + \vec{\omega}_{BA} \times {}^B \vec{v}_{yx} = {}^B \vec{a}_{yx} + \vec{\omega}_{BA} \times {}^B \vec{v}_{yx}, \quad (468)$$

and the last term uses $\widehat{B}^A \vec{p}_{yx} = {}^B \vec{v}_{yx} + \vec{\omega}_{BA} \times \vec{p}_{yx}$ from (455). Substituting and noting that the frame derivative of $\vec{\omega}_{BA}$ is the same in either frame (Section 8.2), so $\widehat{B}^A \vec{\omega}_{BA} = \widehat{B}^B \vec{\omega}_{BA} =: \vec{\alpha}_{BA}$:

$${}^A \vec{a}_{yx} = {}^B \vec{a}_{yx} + \underbrace{2\vec{\omega}_{BA} \times {}^B \vec{v}_{yx}}_{\text{Coriolis}} + \underbrace{\vec{\omega}_{BA} \times (\vec{\omega}_{BA} \times \vec{p}_{yx})}_{\text{centripetal}} + \underbrace{\vec{\alpha}_{BA} \times \vec{p}_{yx}}_{\text{Euler}}. \quad (469)$$

The three correction terms are standard: the *Coriolis* term $2\vec{\omega}_{BA} \times {}^B \vec{v}_{yx}$ couples the body-frame velocity to the frame's rotation, the *centripetal* term $\vec{\omega}_{BA} \times (\vec{\omega}_{BA} \times \vec{p}_{yx})$ points radially inward for steady rotation, and the *Euler* term $\vec{\alpha}_{BA} \times \vec{p}_{yx}$ accounts for changes in the angular velocity itself. Resolving (469) in F_A gives the component form:

$$\begin{aligned} \mathbf{a}_{yx}^A &= \mathbf{R}_B^A \mathbf{a}_{yx}^B + 2\Omega_{BA}^A \mathbf{R}_B^A \mathbf{v}_{yx}^B + \Omega_{BA}^A \Omega_{BA}^A \mathbf{p}_{yx}^A + \dot{\Omega}_{BA}^A \mathbf{p}_{yx}^A \\ &= {}^B \mathbf{a}_{yx}^A + 2\Omega_{BA}^A \mathbf{R}_B^A \mathbf{v}_{yx}^B + \Omega_{BA}^A \Omega_{BA}^A \mathbf{p}_{yx}^A + \dot{\Omega}_{BA}^A \mathbf{p}_{yx}^A. \end{aligned} \quad (470)$$

The same identity arises from differentiating the component-vector relationship in (457) and substituting $\dot{\mathbf{R}}_B^A = \mathbf{R}_B^A \Omega_{BA}^A$ from (444):

$$\begin{aligned} \mathbf{a}_{yx}^A &= \frac{d}{dt} (\mathbf{v}_{yx}^A) = \frac{d}{dt} ({}^B \mathbf{v}_{yx}^A + \Omega_{BA}^A \mathbf{p}_{yx}^A) \\ &= \frac{d}{dt} (\mathbf{R}_B^A \mathbf{v}_{yx}^B + \Omega_{BA}^A \mathbf{p}_{yx}^A) \\ &= \mathbf{R}_B^A \dot{\mathbf{v}}_{yx}^B + \dot{\mathbf{R}}_B^A \mathbf{v}_{yx}^B + \Omega_{BA}^A \dot{\mathbf{p}}_{yx}^A + \dot{\Omega}_{BA}^A \mathbf{p}_{yx}^A \\ &= \mathbf{R}_B^A \mathbf{a}_{yx}^B + \mathbf{R}_B^A \Omega_{BA}^A \mathbf{v}_{yx}^B + \Omega_{BA}^A \mathbf{v}_{yx}^A + \dot{\Omega}_{BA}^A \mathbf{p}_{yx}^A \\ &= {}^B \mathbf{a}_{yx}^A + \mathbf{R}_B^A \mathbf{R}_A^B \Omega_{BA}^A \mathbf{R}_B^A \mathbf{v}_{yx}^B + \Omega_{BA}^A \mathbf{v}_{yx}^A + \dot{\Omega}_{BA}^A \mathbf{p}_{yx}^A \\ &= {}^B \mathbf{a}_{yx}^A + \Omega_{BA}^A \mathbf{R}_B^A \mathbf{v}_{yx}^B + \Omega_{BA}^A [{}^B \mathbf{v}_{yx}^A + \Omega_{BA}^A \mathbf{p}_{yx}^A] + \dot{\Omega}_{BA}^A \mathbf{p}_{yx}^A \\ &= {}^B \mathbf{a}_{yx}^A + \Omega_{BA}^A \mathbf{R}_B^A \mathbf{v}_{yx}^B + \Omega_{BA}^A [\mathbf{R}_B^A \mathbf{v}_{yx}^B + \Omega_{BA}^A \mathbf{p}_{yx}^A] + \dot{\Omega}_{BA}^A \mathbf{p}_{yx}^A \\ &= {}^B \mathbf{a}_{yx}^A + 2\Omega_{BA}^A \mathbf{R}_B^A \mathbf{v}_{yx}^B + \Omega_{BA}^A \Omega_{BA}^A \mathbf{p}_{yx}^A + \dot{\Omega}_{BA}^A \mathbf{p}_{yx}^A. \end{aligned} \quad (471)$$

Note here that ${}^B \mathbf{v}_{yx}^A$ is the velocity of y relative to x as seen from frame F_B resolve in frame F_A . In the common special case where y is fixed in the body frame ($\dot{\mathbf{p}}_{yx}^B = \mathbf{0}$ and $\ddot{\mathbf{p}}_{yx}^B = \mathbf{0}$), the Coriolis term vanishes and the inertial acceleration reduces to a pure centripetal and Euler contribution—the familiar rigid-body result.

16 Discrete Time Systems

Digital state estimators operate on samples taken at discrete times $t_k = k\Delta t$, $k \in \mathbb{Z}^+$, with fixed discrete time intervals $\Delta t > 0$. Hence, the continuous time differential equations of Section 15 need to be integrated into difference equations for discrete time intervals Δt .

For the translational properties, standard numerical integration schemes in \mathbb{R}^3 apply directly. For the rotational subsystem, the state lives on $SO(3)$ (or \mathbb{H}), and naive Euler integration would push the result off the manifold. Hence, we need to use the integration methods using exponential maps as developed in Section 14.

16.1 Attitude

16.1.1 Zeroth-Order Hold

Assume that the angular velocity is held constant at $\boldsymbol{\omega}_{BA}^B(t_{k+1})$ over the interval $[t_k, t_{k+1}]$. The discrete time update equation is given in (437) as:

$$\begin{aligned} \mathbf{R}_B^A(t_{k+1}) &= \mathbf{R}_B^A(t_k) \text{Exp}(\boldsymbol{\omega}_{BA}^B(t_k)\Delta t) \\ &= \mathbf{R}_B^A(t_k) \exp([\boldsymbol{\omega}_{BA}^B(t_k)\Delta t]_{\times}), \end{aligned} \quad (472)$$

where the matrix exponential is evaluated in closed form by *Rodrigues'* formula (346).

Using the left-multiplied form (444) with the angular velocity resolved in F_A , the same update is:

$$\mathbf{R}_B^A(t_{k+1}) = \text{Exp}(\boldsymbol{\omega}_{BA}^A(t_{k+1})\Delta t) \mathbf{R}_B^A(t_k), \quad (473)$$

with $\boldsymbol{\omega}_{BA}^A(t_{k+1}) = \mathbf{R}_B^A(t_k) \boldsymbol{\omega}_{BA}^B(t_k)$ via the adjoint action (Section 10.5.1).

For unit quaternion representations, the quaternion ODE (445) in closed form over a zeroth-order hold interval as:

$$\mathbf{q}_B^A(t_{k+1}) = \mathbf{q}_B^A(t_k) \otimes \text{Exp}(\boldsymbol{\omega}_{BA}^B(t_k)\Delta t). \quad (474)$$

Since both $\mathbf{q}_B^A(t_k)$ and $\text{Exp}(\boldsymbol{\omega}_{BA}^B(t_k)\Delta t)$ are unit quaternions and quaternion multiplication preserves the norm, the updated quaternion $\mathbf{q}_B^A(t_{k+1})$ remains on \mathbb{H} with unit norm in exact arithmetic. In finite-precision arithmetic, small numerical drift may accumulate, and periodic re-normalization $\mathbf{q}_B^A \leftarrow \mathbf{q}_B^A / \|\mathbf{q}_B^A\|$ is advisable.

For small rotation increments ($\|\boldsymbol{\omega}_{BA,k}^B \Delta t\| \ll 1$), the first-order Taylor expansion of the quaternion exponential yields:

$$\mathbf{q}_B^A(t_{k+1}) \approx \mathbf{q}_B^A(t_k) \otimes [1, \frac{1}{2} \boldsymbol{\omega}_{BA}^B(t_k)\Delta t]^T, \quad (475)$$

which is the small-angle quaternion update commonly seen in attitude filters.

16.1.2 First-Order Hold

The first-order integration from (441):

$$\mathbf{R}_B^A(t_{k+1}) = \mathbf{R}_B^A(t_k) \text{Exp}\left(\frac{\boldsymbol{\omega}_{BA}^B(t_k) + \boldsymbol{\omega}_{BA}^B(t_{k+1})}{2} \Delta t\right). \quad (476)$$

This is exact when $\boldsymbol{\omega}_{BA}^B(t)$ varies linearly on $[t_k, t_{k+1}]$ and the instantaneous axis of rotation remains constant. When the axis itself rotates within the interval, the median angular rate incurs an $O(\Delta t^3)$ error due to the non-commutativity of $SO(3)$; this error is known as *coning* and is corrected by higher-order schemes that use the right Jacobian \mathbf{J}_r of Section ??:

$$\mathbf{R}_B^A(t_{k+1}) = \mathbf{R}_B^A(t_k) \text{Exp}(\boldsymbol{\omega}_{BA}^B(t_k)\Delta t + \frac{1}{2} \mathbf{J}_r(\boldsymbol{\omega}_{BA}^B(t_k)\Delta t) (\boldsymbol{\omega}_{BA}^B(t_{k+1}) - \boldsymbol{\omega}_{BA}^B(t_k)) \Delta t). \quad (477)$$

16.2 Position

Let F_A be a frame and x, y be points. The translational position satisfies $\dot{\mathbf{p}}_{yx}^A = \mathbf{v}_{yx}^A$, which is an ODE in the vector space \mathbb{R}^3 and admits standard numerical integration schemes directly, without the complications of working on a manifold. The *zeroth-order hold* (Euler) update assumes the velocity is constant over the step:

$$\mathbf{p}_{yx}^A(t_{k+1}) = \mathbf{p}_{yx}^A(t_k) + \mathbf{v}_{yx}^A(t_k)\Delta t. \quad (478)$$

If velocity samples are available at both endpoints, the *first-order hold* (trapezoidal) update is second-order accurate:

$$\mathbf{p}_{yx}^A(t_{k+1}) = \mathbf{p}_{yx}^A(t_k) + \frac{1}{2} (\mathbf{v}_{yx}^A(t_k) + \mathbf{v}_{yx}^A(t_{k+1})) \Delta t. \quad (479)$$

When acceleration is also known over the step, a Taylor expansion to second order in Δt gives the kinematic position update:

$$\mathbf{p}_{yx}^A(t_{k+1}) = \mathbf{p}_{yx}^A(t_k) + \mathbf{v}_{yx}^A(t_k)\Delta t + \frac{1}{2}\mathbf{a}_{yx}^A(t_k)\Delta t^2, \quad (480)$$

which is the standard propagation equation used in strapdown inertial navigation.

16.3 Velocity

The velocity satisfies $\dot{\mathbf{v}}_{yx}^A = \mathbf{a}_{yx}^A$. Because velocity and acceleration both live in \mathbb{R}^3 , the same standard schemes apply. The zeroth-order hold (Euler) update is:

$$\mathbf{v}_{yx}^A(t_{k+1}) = \mathbf{v}_{yx}^A(t_k) + \mathbf{a}_{yx}^A(t_k)\Delta t, \quad (481)$$

and the trapezoidal update is:

$$\mathbf{v}_{yx}^A(t_{k+1}) = \mathbf{v}_{yx}^A(t_k) + \frac{1}{2} (\mathbf{a}_{yx}^A(t_k) + \mathbf{a}_{yx}^A(t_{k+1})) \Delta t. \quad (482)$$

In inertial navigation, the acceleration is typically measured in the body frame by an accelerometer as \mathbf{a}_{BA}^B (with gravity subtracted); in that case the rotation matrix from the attitude subsystem must be applied before the velocity can be integrated in F_A :

$$\mathbf{v}_{yx}^A(t_{k+1}) = \mathbf{v}_{yx}^A(t_k) + \mathbf{R}_B^A(t_{k+1}) \mathbf{a}_{yx}^B(t_{k+1})\Delta t. \quad (483)$$

This coupling—attitude enters the velocity update, and velocity enters the position update—is why the attitude subsystem is propagated first within each time step of a strapdown inertial navigation filter.

16.4 Acceleration

Acceleration is not an independent state in pure kinematics: it is either an input from a dynamics model or a sampled measurement from an accelerometer. At the discrete level, acceleration samples $\mathbf{a}_{yx}^{(\cdot)}(t_k)$ are simply held or interpolated to drive the velocity update (481).

Angular acceleration $\boldsymbol{\alpha}_{yx}^B(t_k) := \dot{\boldsymbol{\omega}}_{yx}^B(t_k)$ enters analogously on the rotational side: it is an input to the first-order-hold attitude update (477) but is not itself propagated as a state.

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